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Advanced Digression: Puppies and Kittens and the Golden Ratio

The problems below will help you to discover the stunning fact that the golden ratio is intimately involved with the Puppies and Kittens game. Let (x_n, y_n) be the Puppies and Kittens oasis satisfying $y_n - x_n = n$. For example,

$$x_1 = 1, y_1 = 2, \quad x_2 = 3, y_2 = 5, \quad x_3 = 4, y_3 = 7.$$

Our goal is to show, for all $n = 1,2,3,\ldots$, that $x_n = \lfloor n\tau \rfloor$, and thus $y_n = \lfloor n(\tau+1) \rfloor$, where τ is the famous, ubiquitous *Golden Ratio*:

$$\tau = \frac{1+\sqrt{5}}{2}.$$

The key idea is a little-known fact called Beatty's Theorem, which you will prove in problems 4 and 5 below.

1 Show that $\tau^2 = \tau + 1$ and, thus,

$$\frac{1}{\tau} + \frac{1}{\tau+1} = 1.$$

2 Two disjoint sets whose union is the natural numbers $\mathbf{N} = \{1,2,3,\ldots\}$ are said to *partition* $\mathbf{N}$. In other words, if A and B partition $\mathbf{N}$, then every natural number is a member of exactly one of the sets A, B. No overlaps, and no omissions.

Let (x_n, y_n) be the Puppies and Kittens oasis satisfying $y_n - x_n = n$. For example,

$$x_1 = 1, y_1 = 2, x_2 = 3, y_2 = 5.$$

Verify that the two sets

$$A = \{x_1, x_2, x_3, \ldots\} \text{ and } B = \{y_1, y_2, y_3, \ldots\}$$

partition the natural numbers.

3 Let α be a positive real number. Define the set of *multiples* of α to be the positive integers

$$\{\lfloor \alpha \rfloor, \lfloor 2\alpha \rfloor, \lfloor 3\alpha \rfloor, \ldots\}.$$

For example, if $\alpha = 2$, then the multiples of α are just the even positive integers. Notice that α need not be an integer, or even rational.

Does there exist α such that the multiples of α are the *odd* positive integers?

4 Suppose that there are two numbers α, β whose sets of multiples partition the natural numbers. In other words, every natural number is equal to the floor of an integer times *exactly one* of α or β, and there are no overlaps.

(a) Prove that both α and β are greater than 1.

(b) Suppose that $1 < \alpha < 1.1$. Show that $\beta \geq 10$.

(c) Prove that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Hint: How many numbers less than or equal to 2010 are multiples of 7? How many are multiples of 11?

(d) Prove that α must be irrational (and hence by (a), β must also be irrational).

5 Prove the converse of the above problem; i.e., if α, β are positive irrational numbers satisfying

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

then the multiples of α and the multiples of β partition the natural numbers.

6 Why does this now show that $x_n = \lfloor n\tau \rfloor$, and thus $y_n = \lfloor n(\tau+1) \rfloor$?