

This Game is Rigged!

(I'll bet you \$5 that if you give me \$10, I'll give you \$20.)

Paul Zeitz

University of San Francisco

May 7, 2017



- 1 Foolproof Dice
- 2 The Putdown Game
- 3 Cat and Mouse
- 4 Puppies and Kittens

Four Weird Dice

Four Weird Dice

A: 0, 0, 4, 4, 4, 4

Four Weird Dice

A: 0, 0, 4, 4, 4, 4

B: 3, 3, 3, 3, 3, 3

Four Weird Dice

A: 0, 0, 4, 4, 4, 4

B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6

Four Weird Dice

A: 0, 0, 4, 4, 4, 4

B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6

D: 1, 1, 1, 5, 5, 5

Four Weird Dice

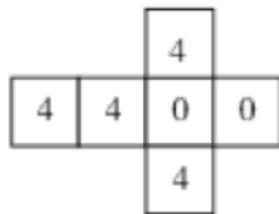
A: 0, 0, 4, 4, 4, 4

B: 3, 3, 3, 3, 3, 3

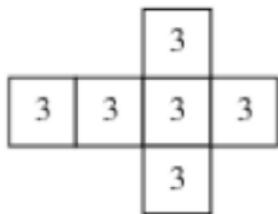
C: 2, 2, 2, 2, 6, 6

D: 1, 1, 1, 5, 5, 5

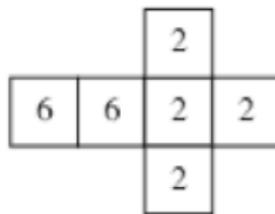
A



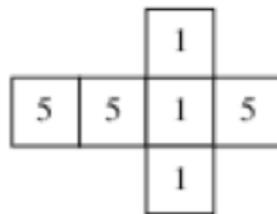
B



C



D



A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

A vs B	0	0	4	4	4	4
3						
3						
3						
3						
3						
3						

A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

A vs B	0	0	4	4	4	4
3	B	B				
3						
3						
3						
3						
3						

A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

A vs B	0	0	4	4	4	4
3	B	B	A	A	A	A
3						
3						
3						
3						
3						

A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

A vs B	0	0	4	4	4	4
3	B	B	A	A	A	A
3	B	B	A	A	A	A
3	B	B	A	A	A	A
3	B	B	A	A	A	A
3	B	B	A	A	A	A
3	B	B	A	A	A	A

A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

A vs B	0	0	4	4	4	4
3	B	B	A	A	A	A
3	B	B	A	A	A	A
3	B	B	A	A	A	A
3	B	B	A	A	A	A
3	B	B	A	A	A	A
3	B	B	A	A	A	A

$$P(\text{A beats B}) = 2/3.$$

A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

B vs. C	2	2	2	2	6	6
3						
3						
3						
3						
3						
3						

A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

B vs. C	2	2	2	2	6	6
3	B	B	B	B		
3	B	B	B	B		
3	B	B	B	B		
3	B	B	B	B		
3	B	B	B	B		
3	B	B	B	B		

A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

B vs. C	2	2	2	2	6	6
3	B	B	B	B	C	C
3	B	B	B	B	C	C
3	B	B	B	B	C	C
3	B	B	B	B	C	C
3	B	B	B	B	C	C
3	B	B	B	B	C	C

A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

B vs. C	2	2	2	2	6	6
3	B	B	B	B	C	C
3	B	B	B	B	C	C
3	B	B	B	B	C	C
3	B	B	B	B	C	C
3	B	B	B	B	C	C
3	B	B	B	B	C	C

$$P(\text{B beats C}) = 2/3.$$

A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

C vs. D	2	2	2	2	6	6
1						
1						
1						
5						
5						
5						

A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

C vs. D	2	2	2	2	6	6
1	C	C	C	C	C	C
1	C	C	C	C	C	C
1	C	C	C	C	C	C
5					C	C
5					C	C
5					C	C

A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

C vs. D	2	2	2	2	6	6
1	C	C	C	C	C	C
1	C	C	C	C	C	C
1	C	C	C	C	C	C
5	D	D	D	D	C	C
5	D	D	D	D	C	C
5	D	D	D	D	C	C

A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

C vs. D	2	2	2	2	6	6
1	C	C	C	C	C	C
1	C	C	C	C	C	C
1	C	C	C	C	C	C
5	D	D	D	D	C	C
5	D	D	D	D	C	C
5	D	D	D	D	C	C

$$P(\text{C beats D}) = 2/3.$$

A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

A vs. D	1	1	1	5	5	5
0						
0						
4						
4						
4						
4						

A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

A vs. D	1	1	1	5	5	5
0	D	D	D	D	D	D
0	D	D	D	D	D	D
4					D	D
4					D	D
4					D	D
4					D	D

A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

A vs. D	1	1	1	5	5	5
0	D	D	D	D	D	D
0	D	D	D	D	D	D
4	A	A	A	D	D	D
4	A	A	A	D	D	D
4	A	A	A	D	D	D
4	A	A	A	D	D	D

A: 0, 0, 4, 4, 4, 4 B: 3, 3, 3, 3, 3, 3

C: 2, 2, 2, 2, 6, 6 D: 1, 1, 1, 5, 5, 5

A vs. D	1	1	1	5	5	5
0	D	D	D	D	D	D
0	D	D	D	D	D	D
4	A	A	A	D	D	D
4	A	A	A	D	D	D
4	A	A	A	D	D	D
4	A	A	A	D	D	D

$$P(\text{D beats A}) = 2/3.$$

So, with probabilities of $2/3$,

So, with probabilities of $2/3$,

A beats B,

So, with probabilities of $2/3$,

A beats B, B beats C,

So, with probabilities of $2/3$,

A beats B, B beats C, C beats D,

So, with probabilities of $2/3$,

A beats B, B beats C, C beats D, and D beats A!

Non-Transitive Bingo!

A

1	2
3	4

B

2	4
5	6

C

1	3
4	5

D

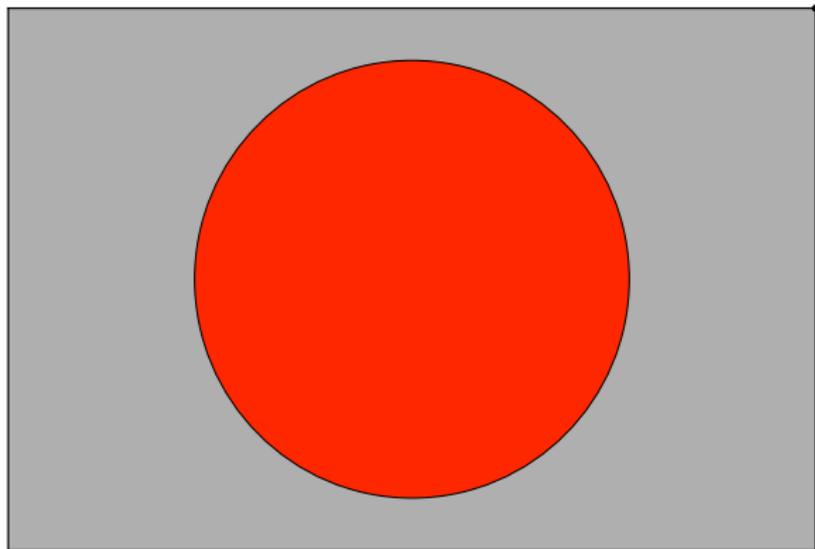
1	5
2	6

The Putdown Game

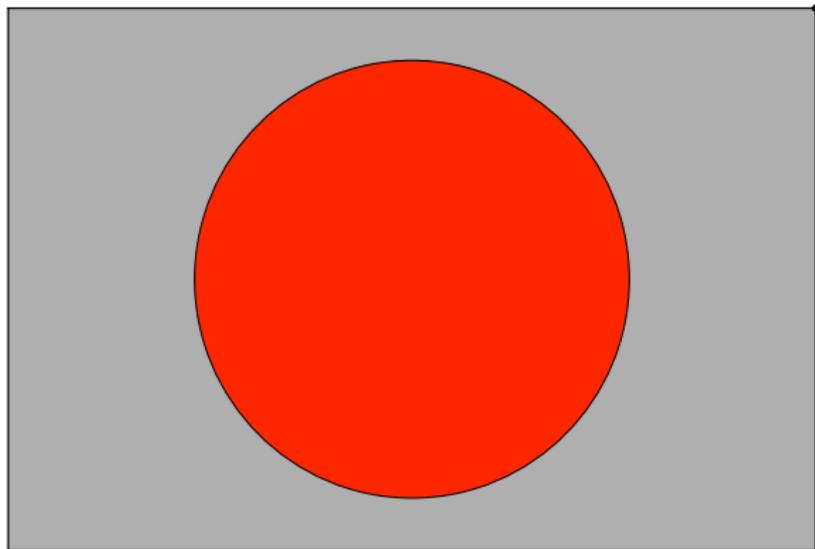
- Two players, A and B, alternate turns placing a penny on the surface of a rectangular table.
- No penny can touch a penny that is already on the table. The table starts out completely bare.
- No penny can touch a penny that is already on the table.
- The player who places the last penny wins.

Try a small playing field.

Try a small playing field.



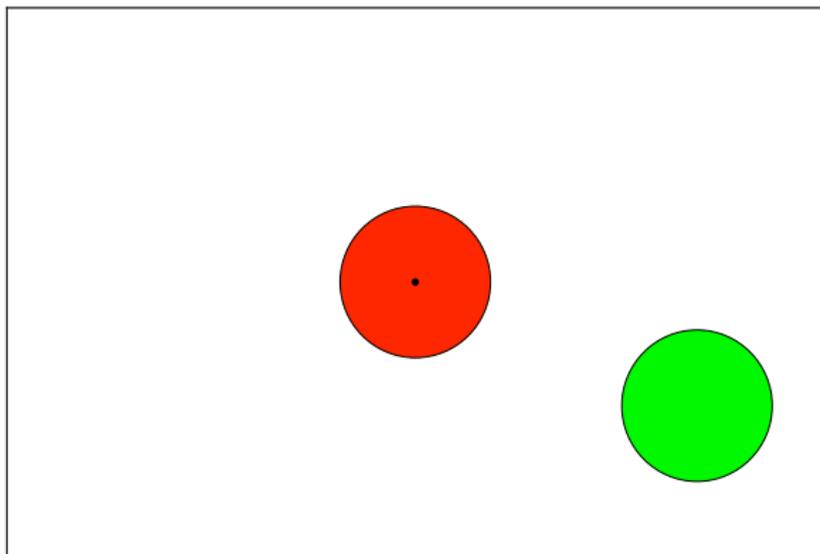
Try a small playing field.



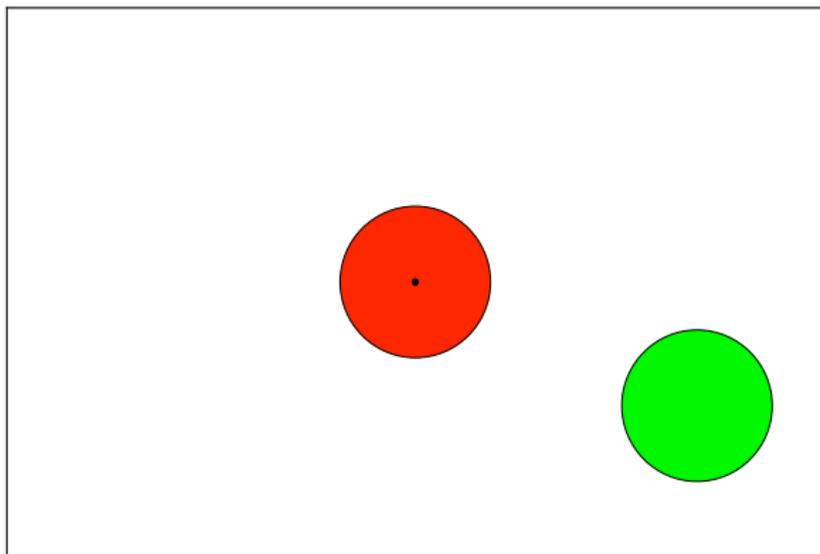
Player A wins!

Try a larger playing field.

Try a larger playing field.

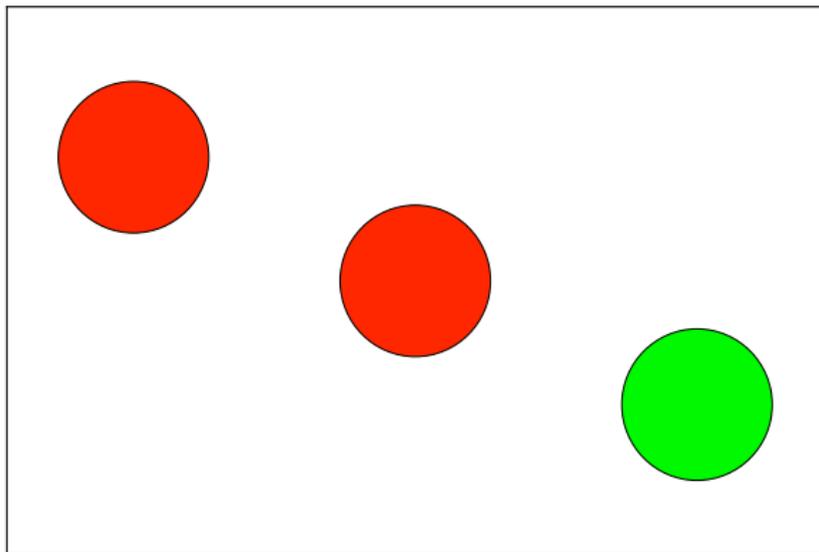


Try a larger playing field.

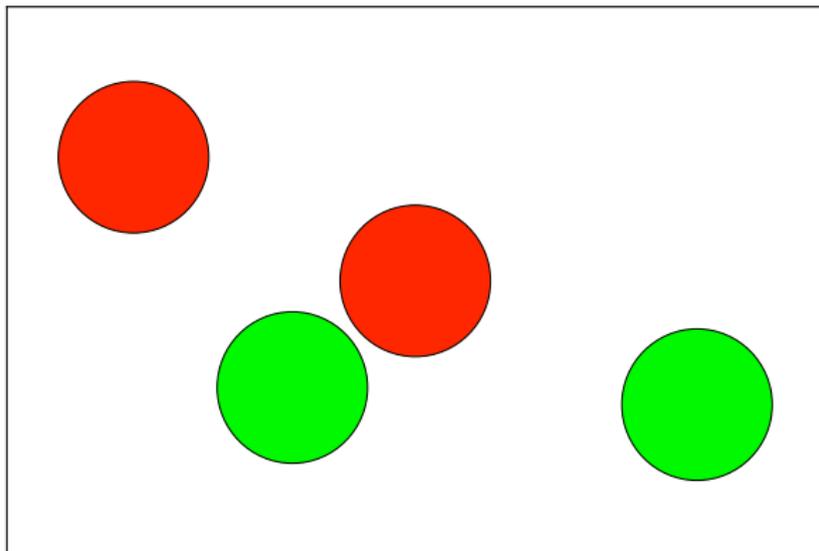


Do you see a strategy?

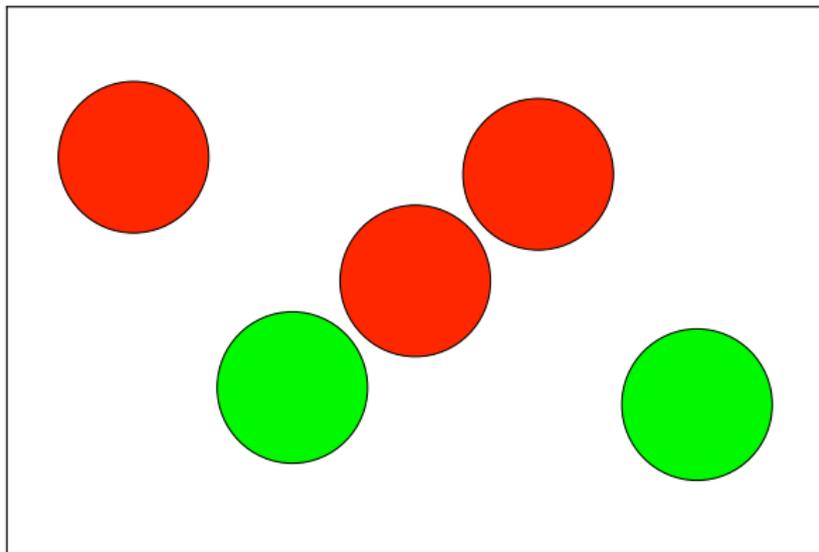
Mirror!



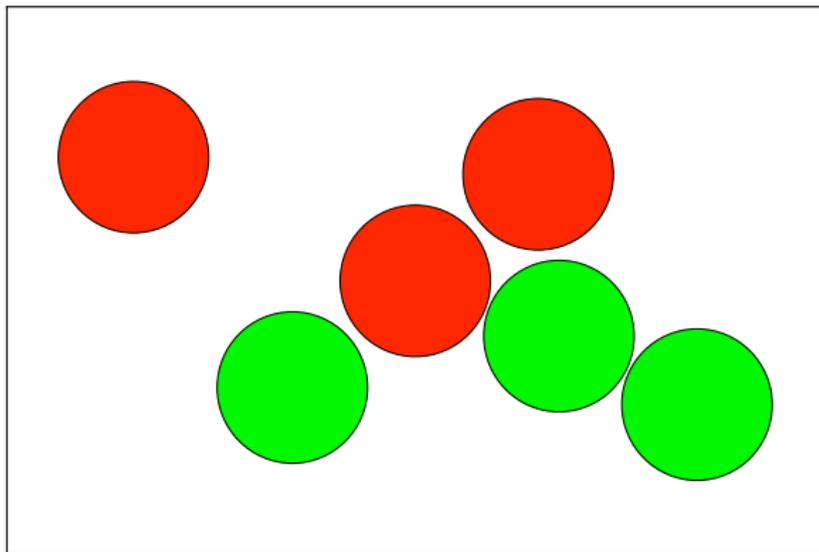
Mirror!



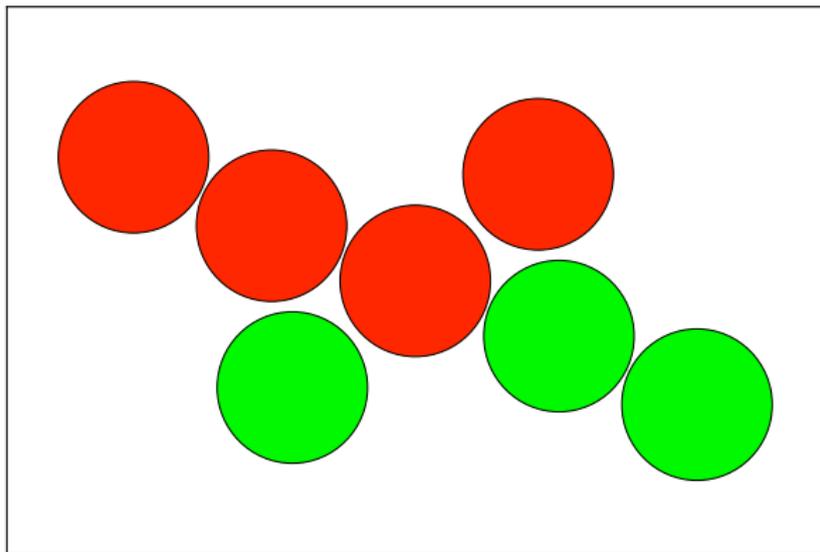
Mirror!



Mirror!

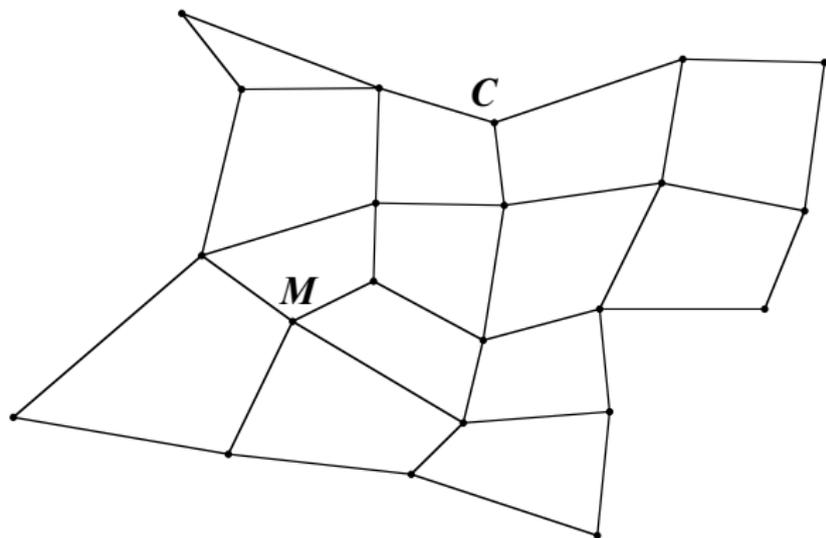


Mirror!



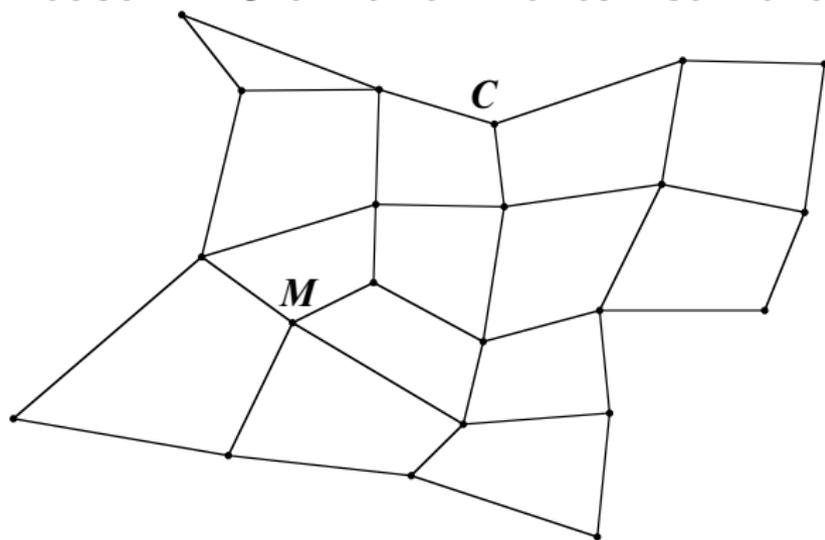
Cat and Mouse (Ravi Vakil, Stanford)

A very polite cat chases an equally polite mouse. They take turns moving on the grid depicted below. Initially, the cat is at the point labeled C ; the mouse is at M .

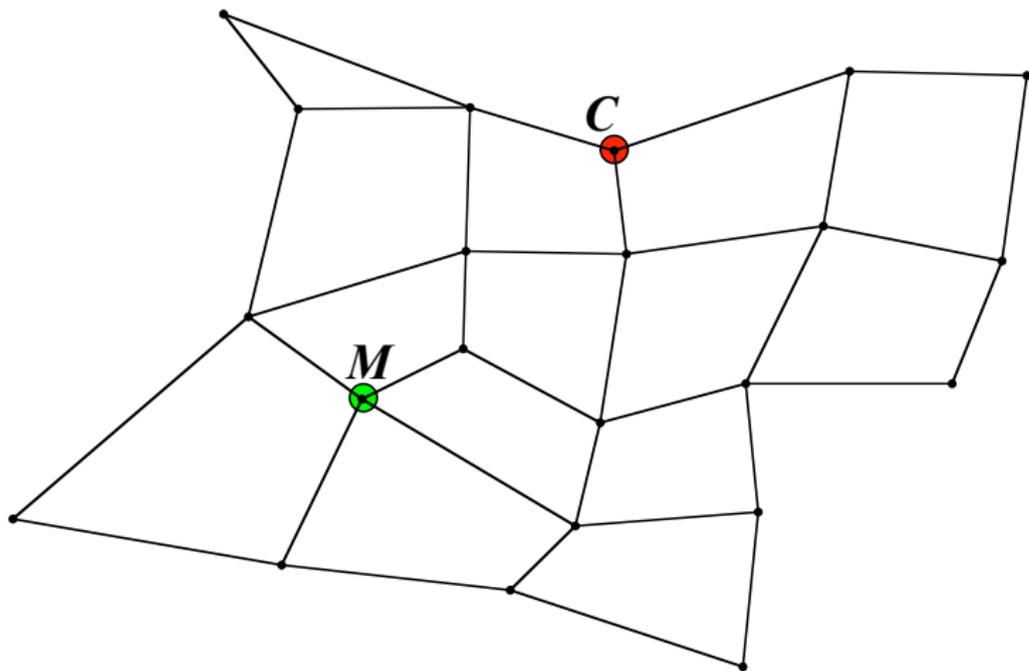


Cat and Mouse

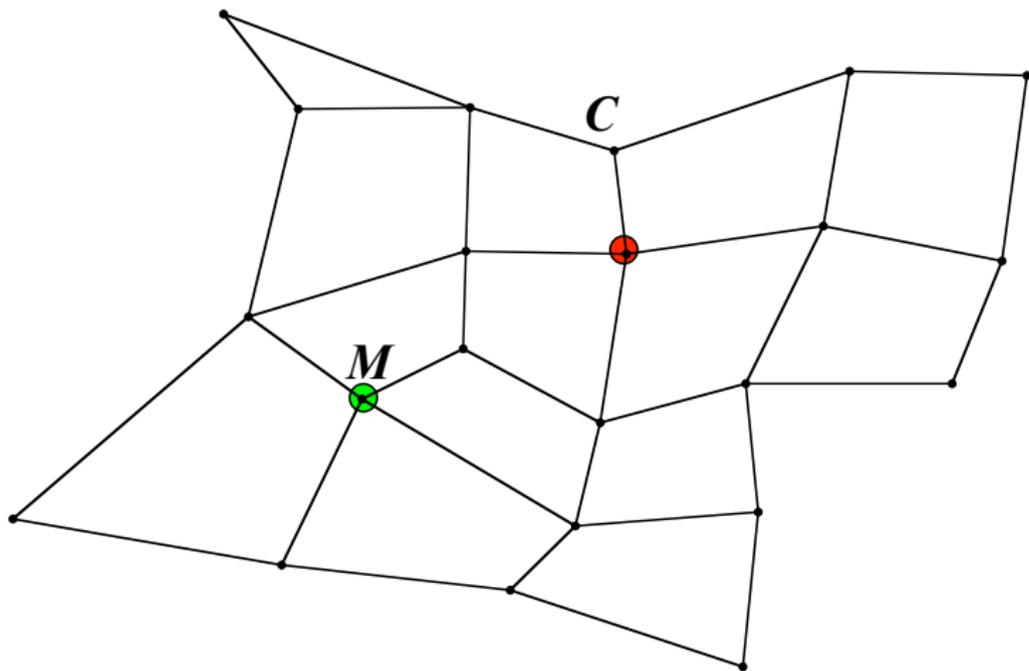
The cat goes first, and can move to any neighboring point connected to it by a single edge. The cat wins if it can reach the mouse in 15 or fewer moves. Can the cat win?



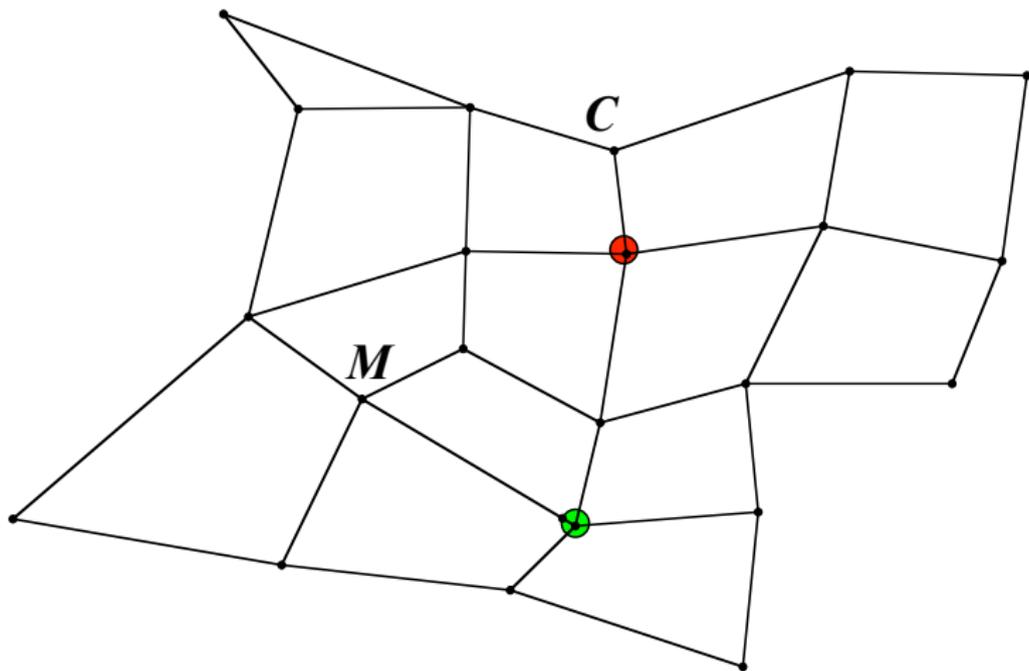
Cat and Mouse



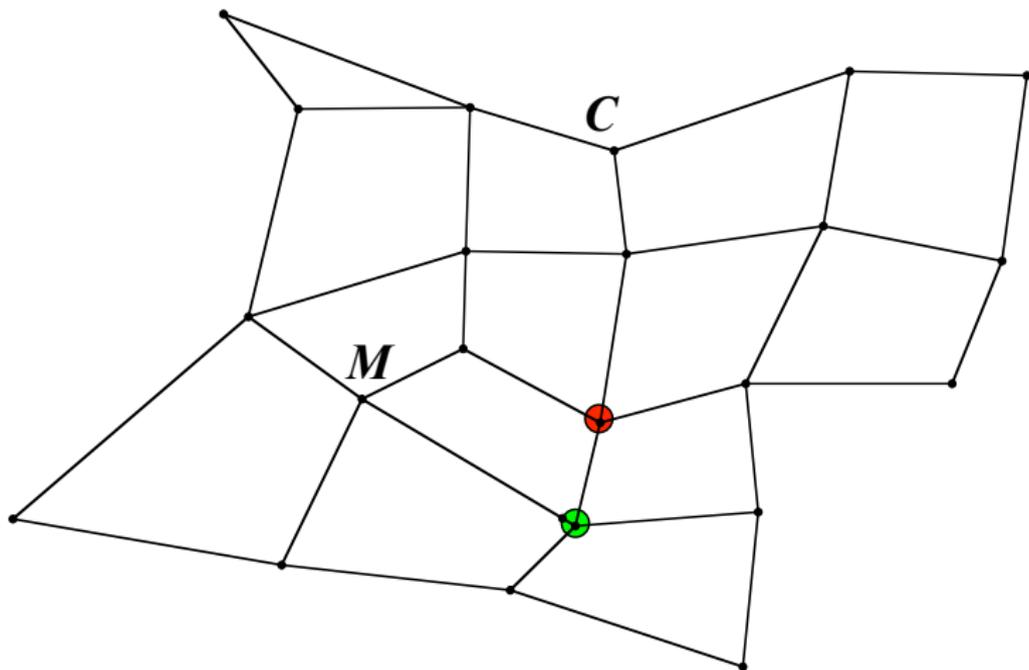
Cat and Mouse



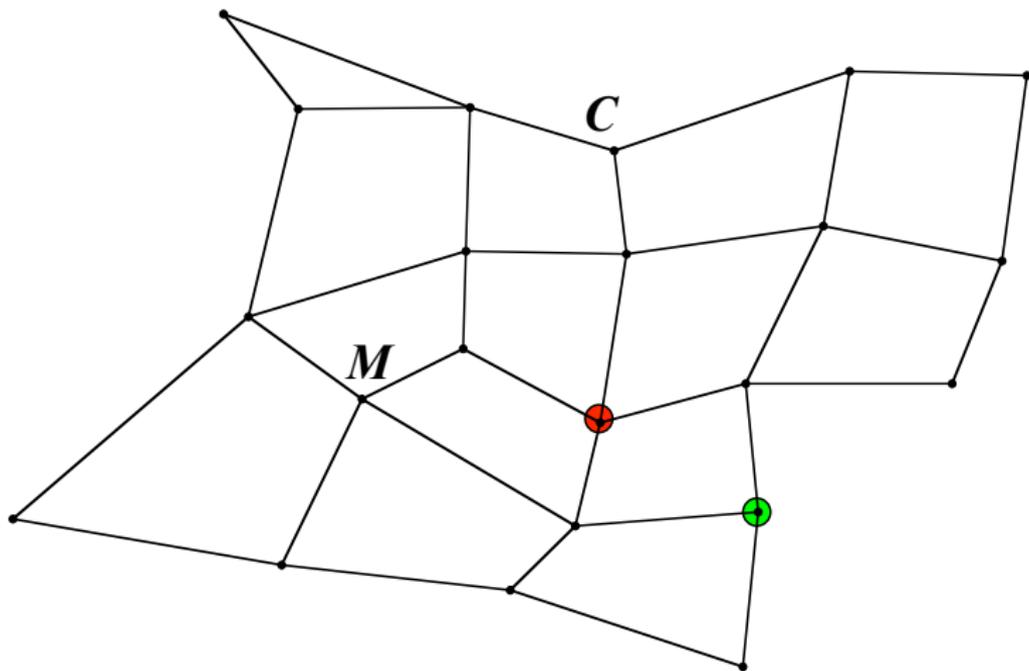
Cat and Mouse



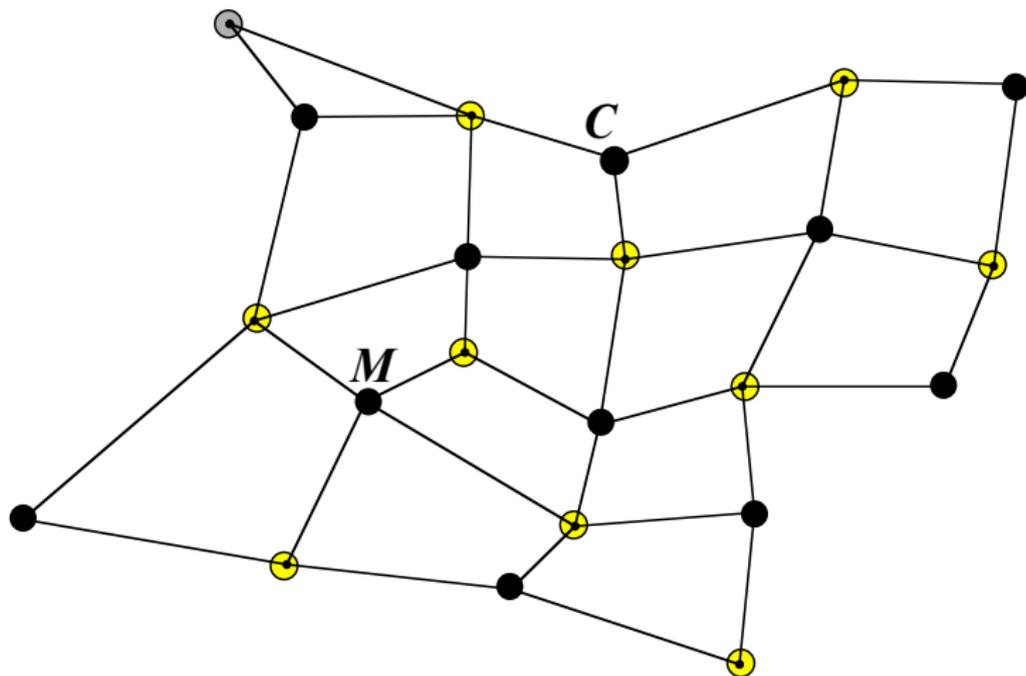
Cat and Mouse



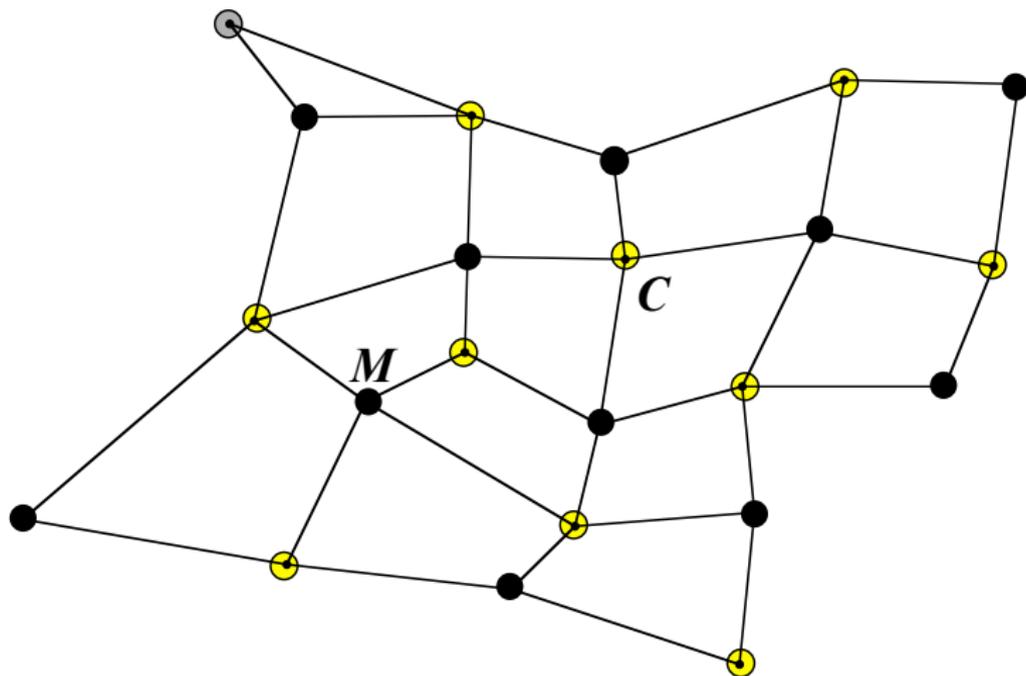
Cat and Mouse



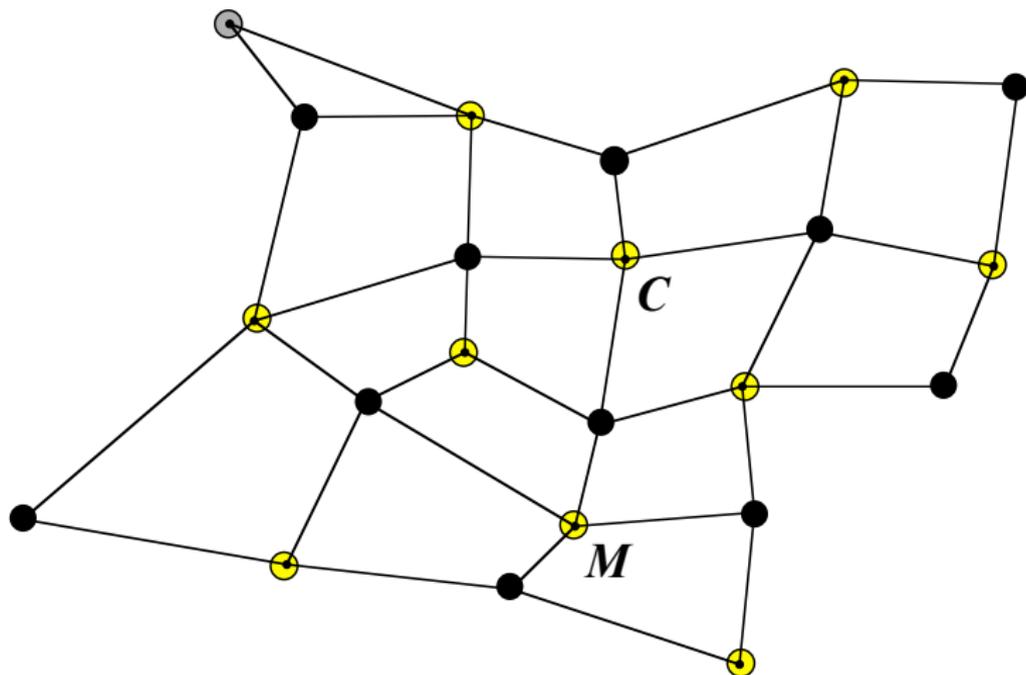
Coloring comes to the rescue!



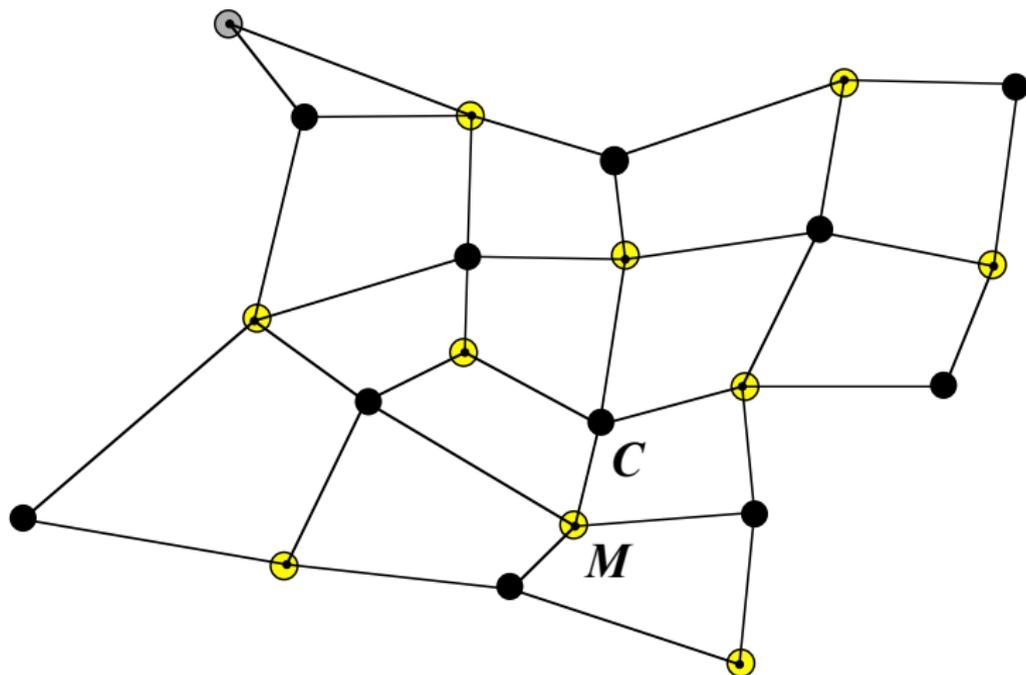
Coloring comes to the rescue!



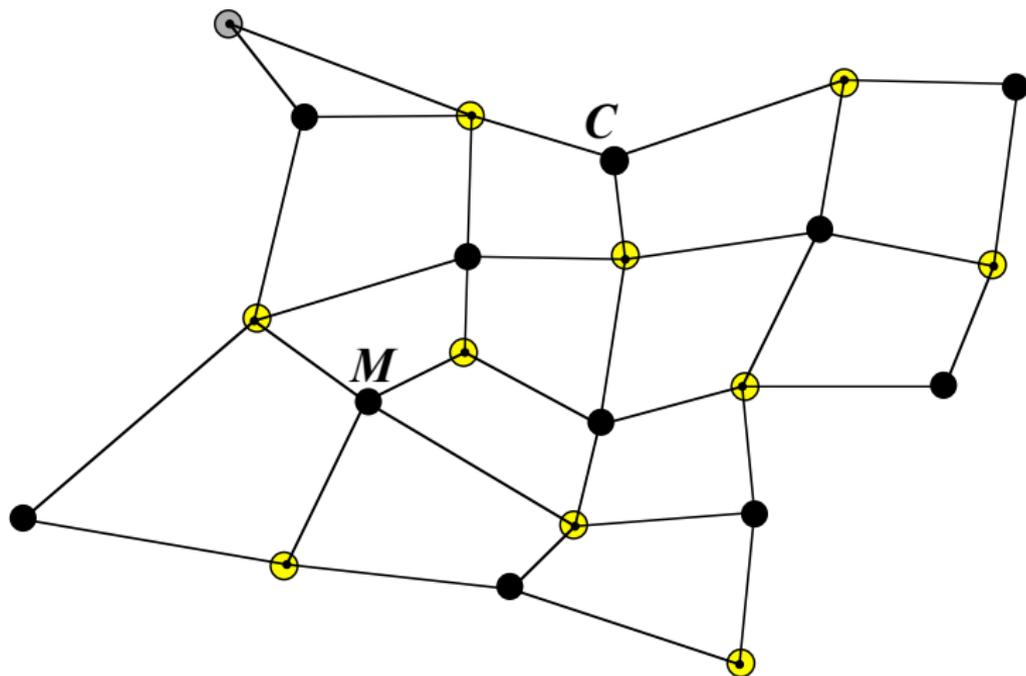
Coloring comes to the rescue!



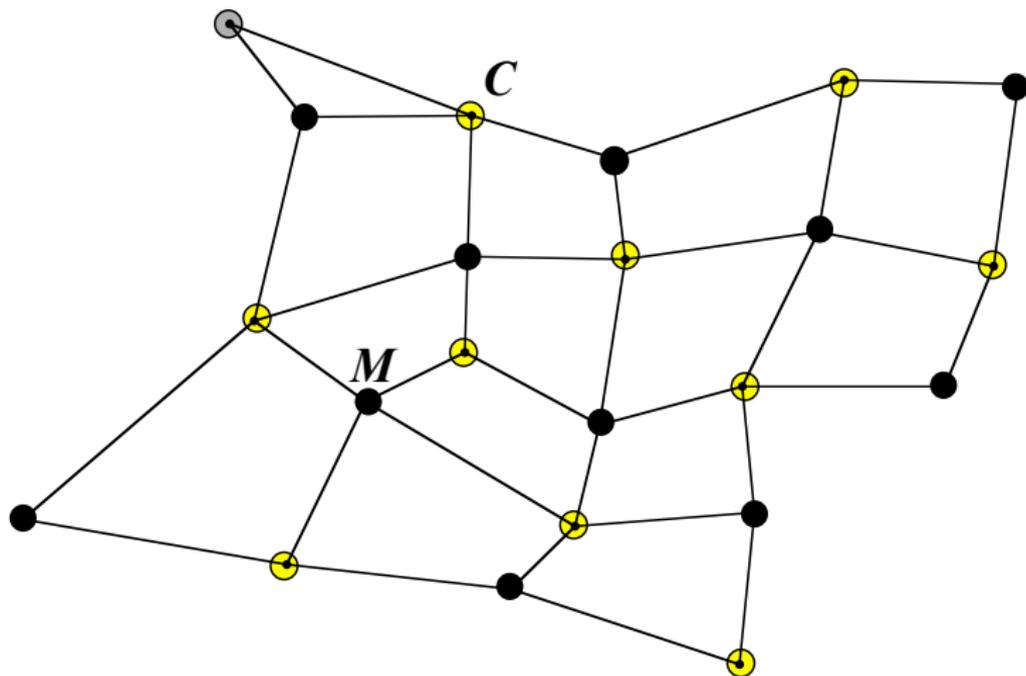
Coloring comes to the rescue!



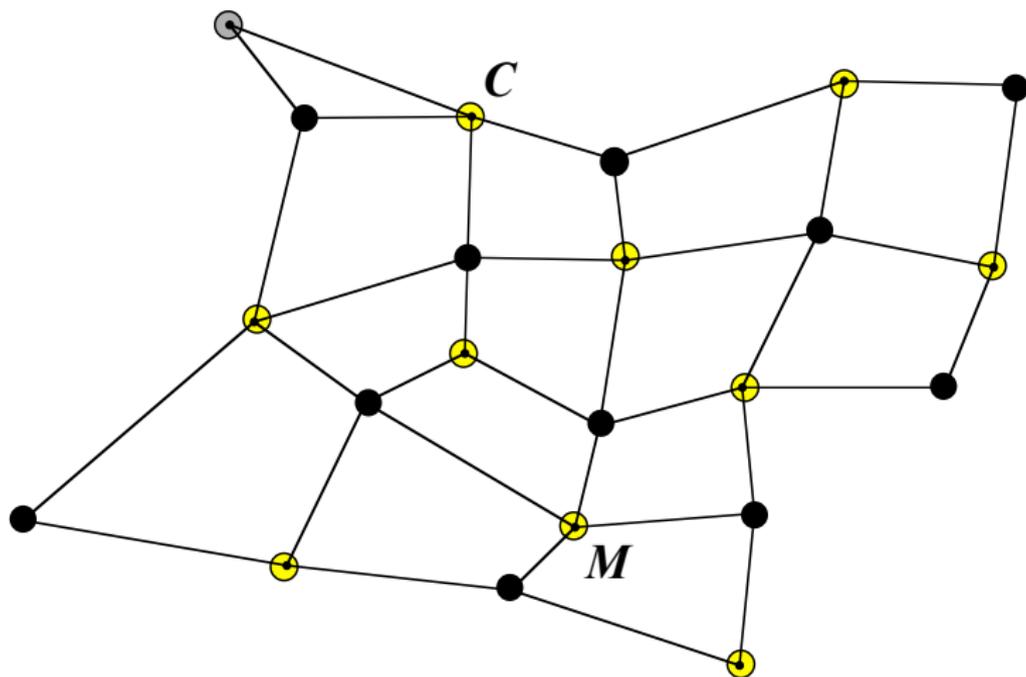
Cat seizes tempo!



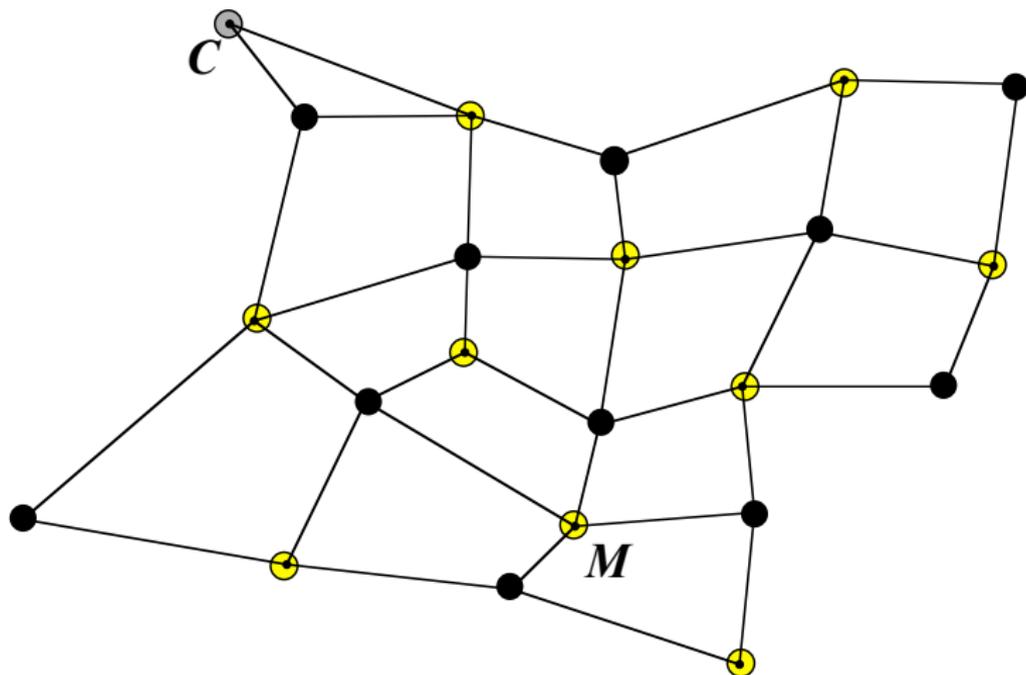
Cat seizes tempo!



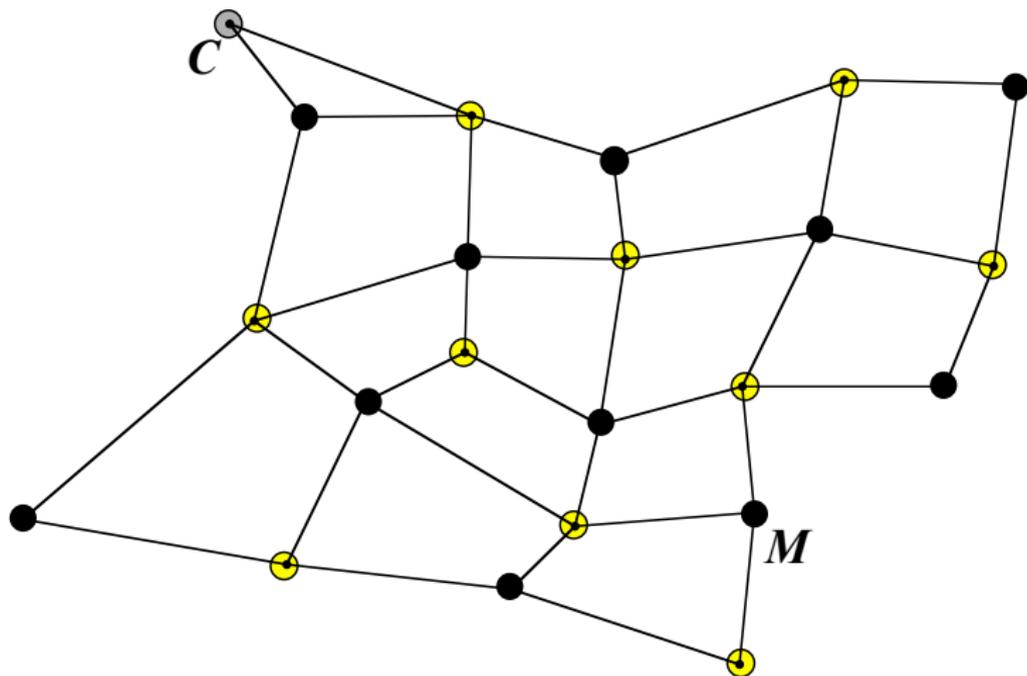
Cat seizes tempo!



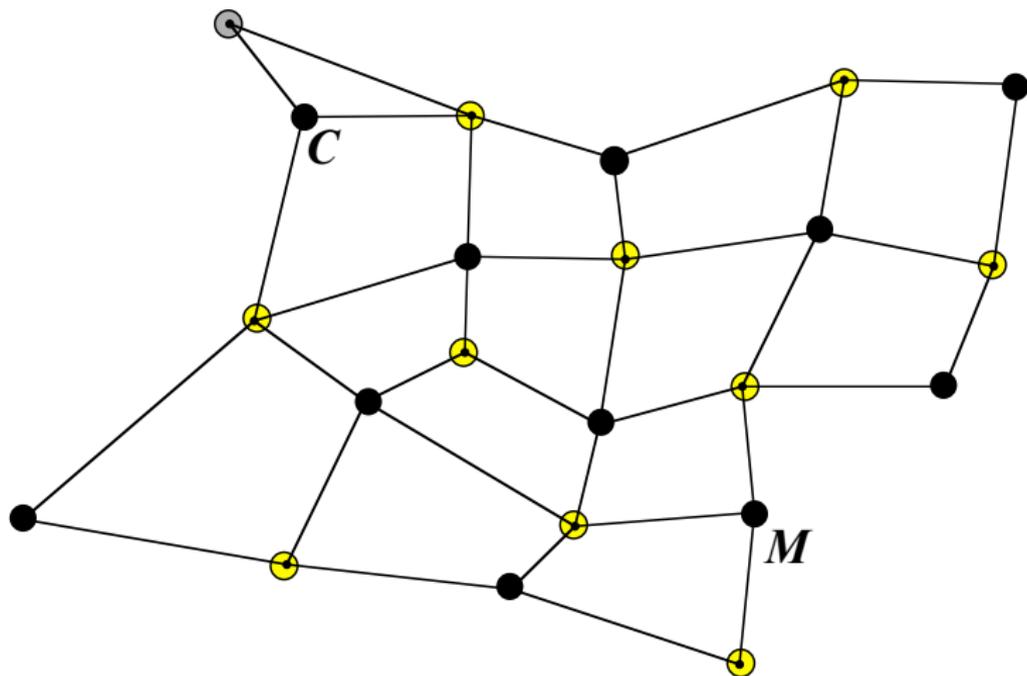
Cat seizes tempo!



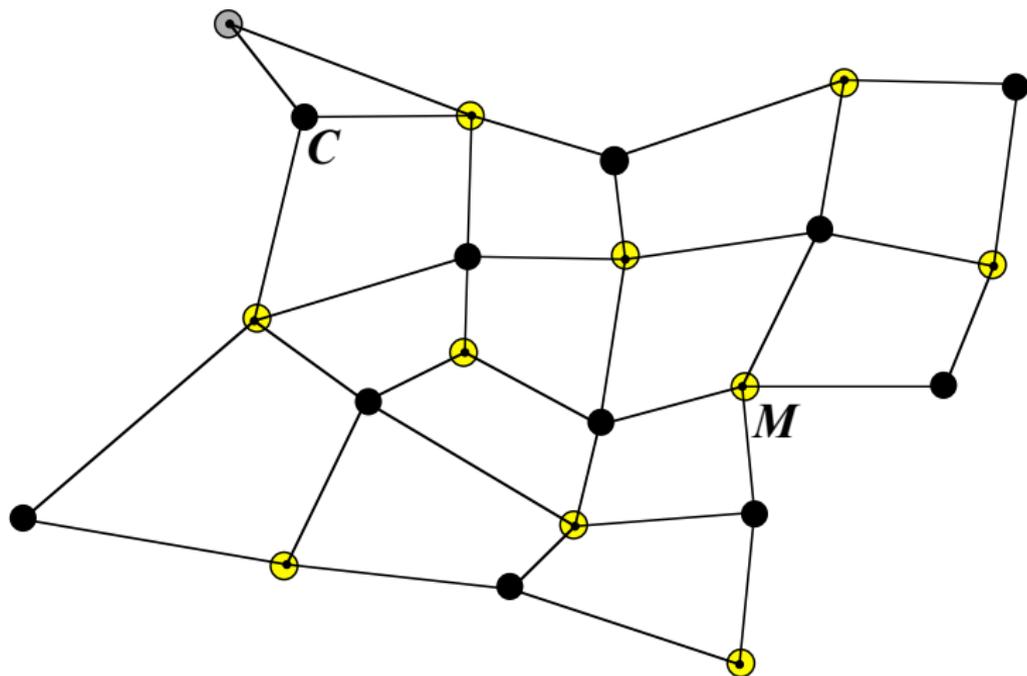
Cat seizes tempo!



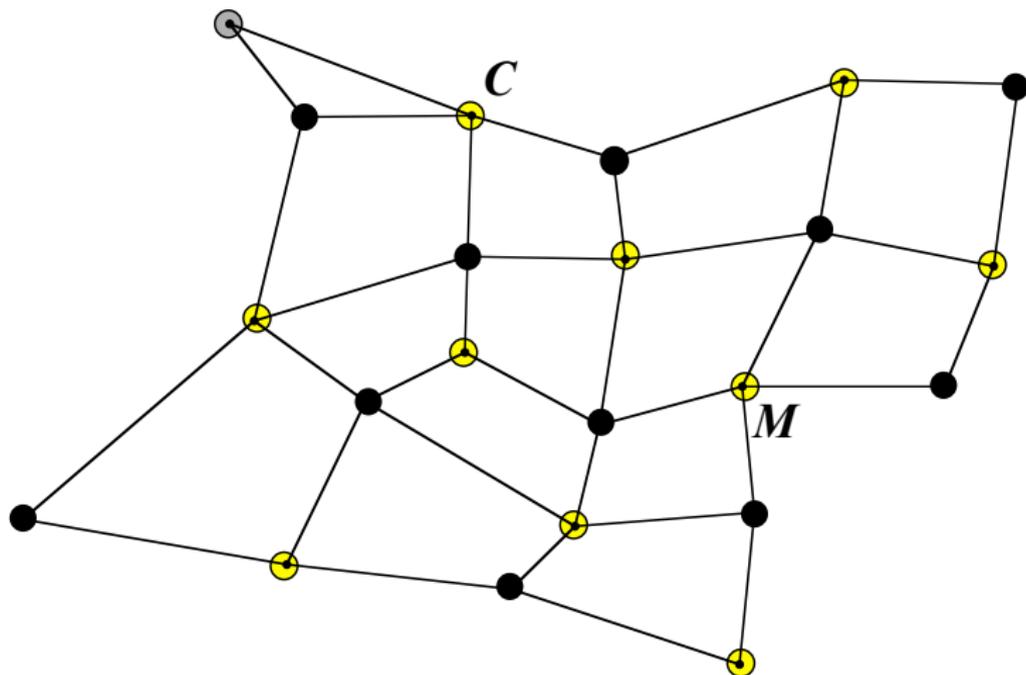
Cat seizes tempo!



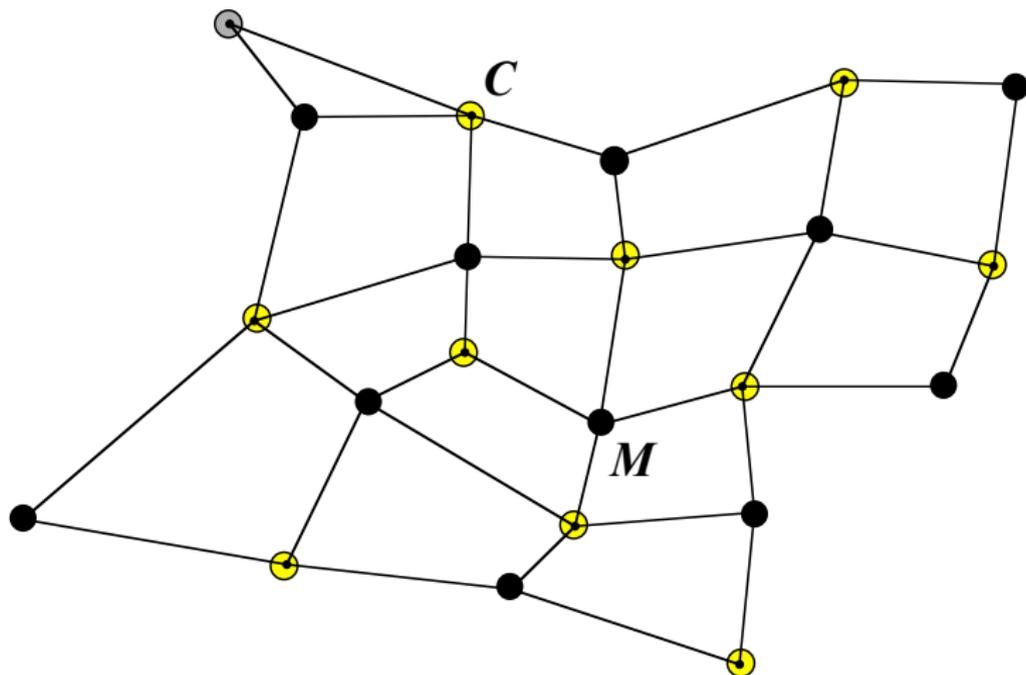
Cat seizes tempo!



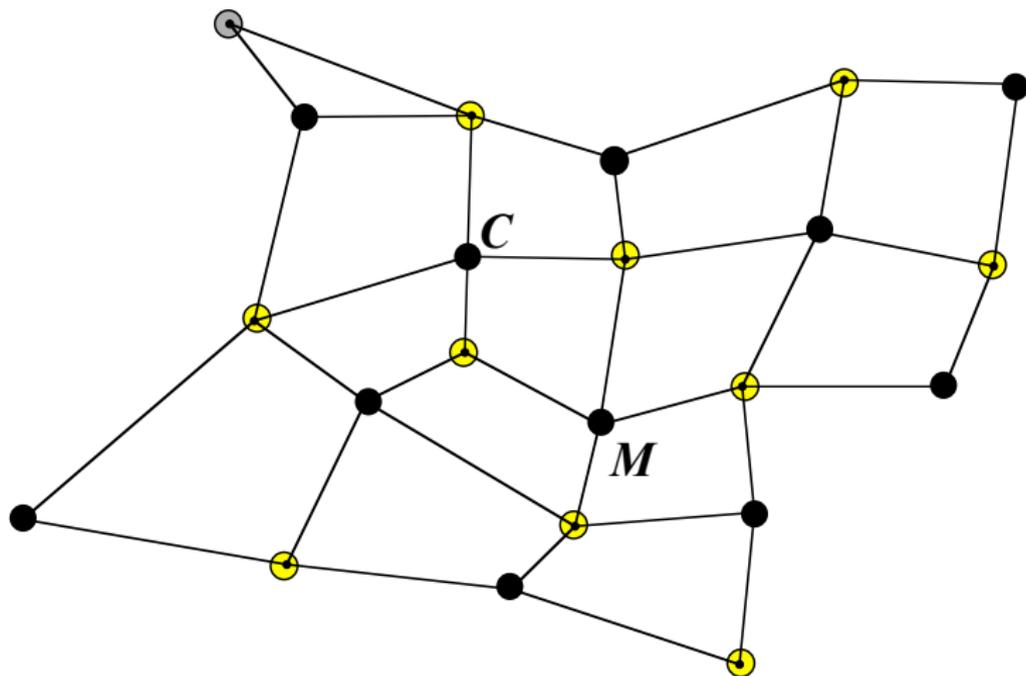
Cat seizes tempo!



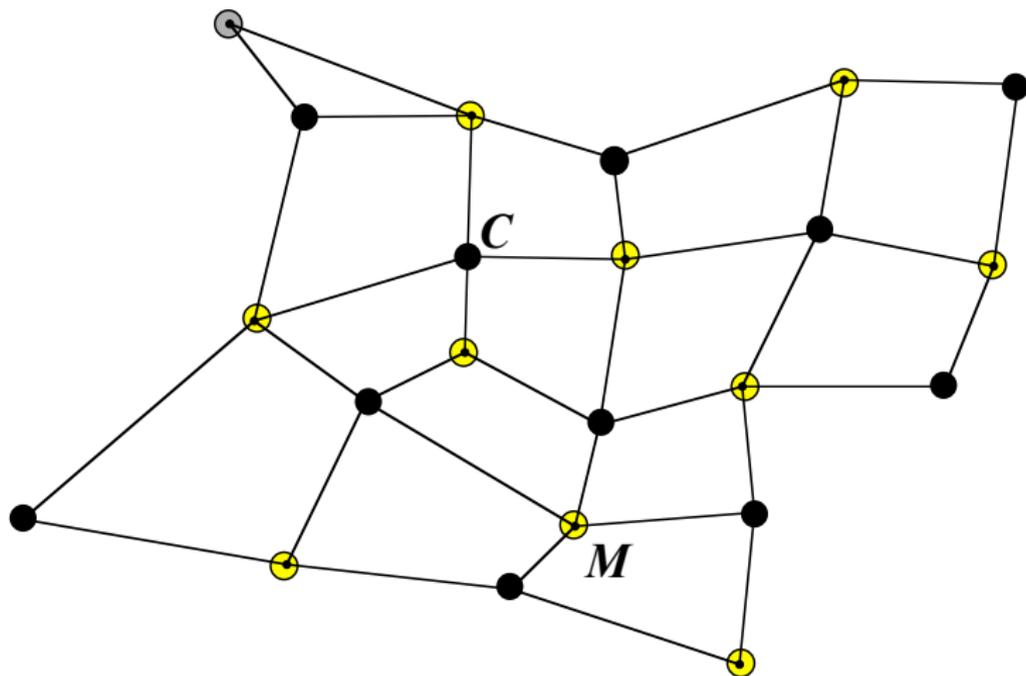
Cat seizes tempo!



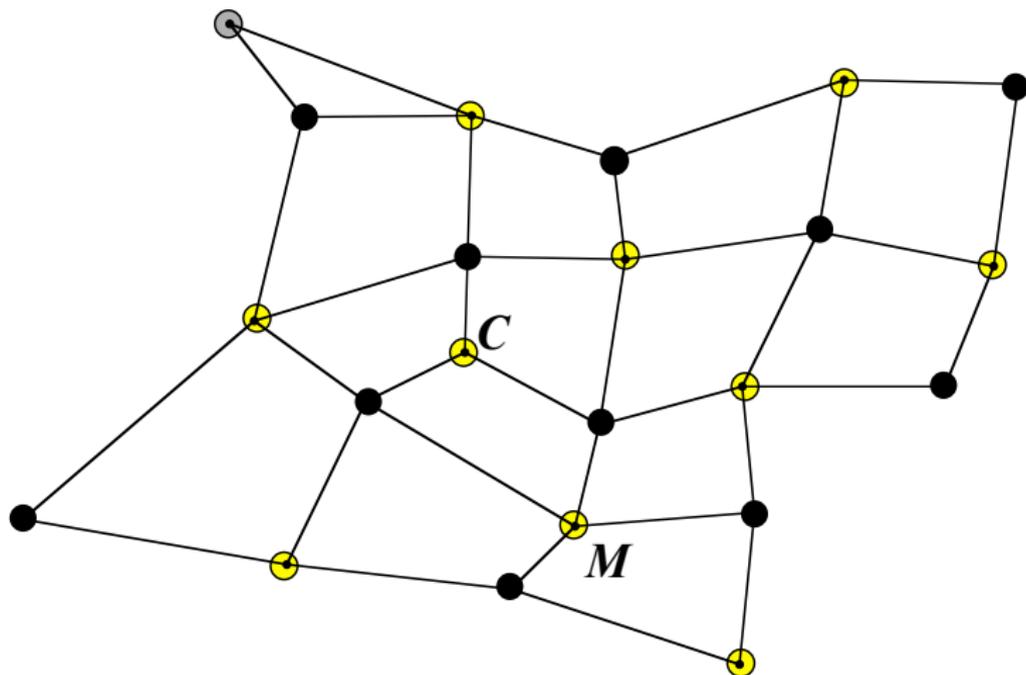
Cat seizes tempo!



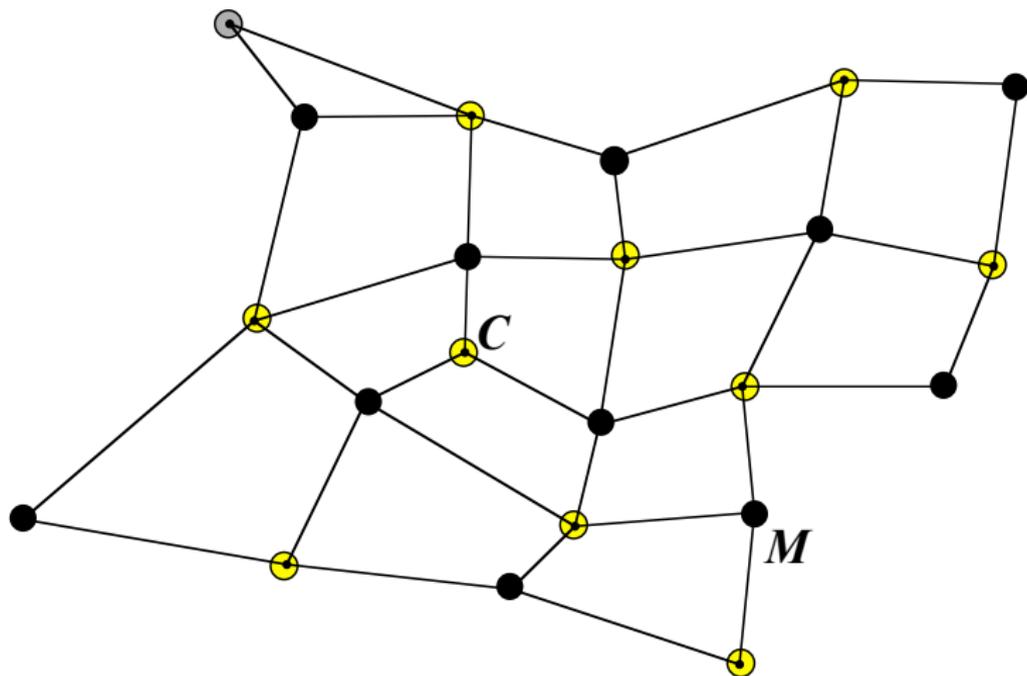
Cat seizes tempo!



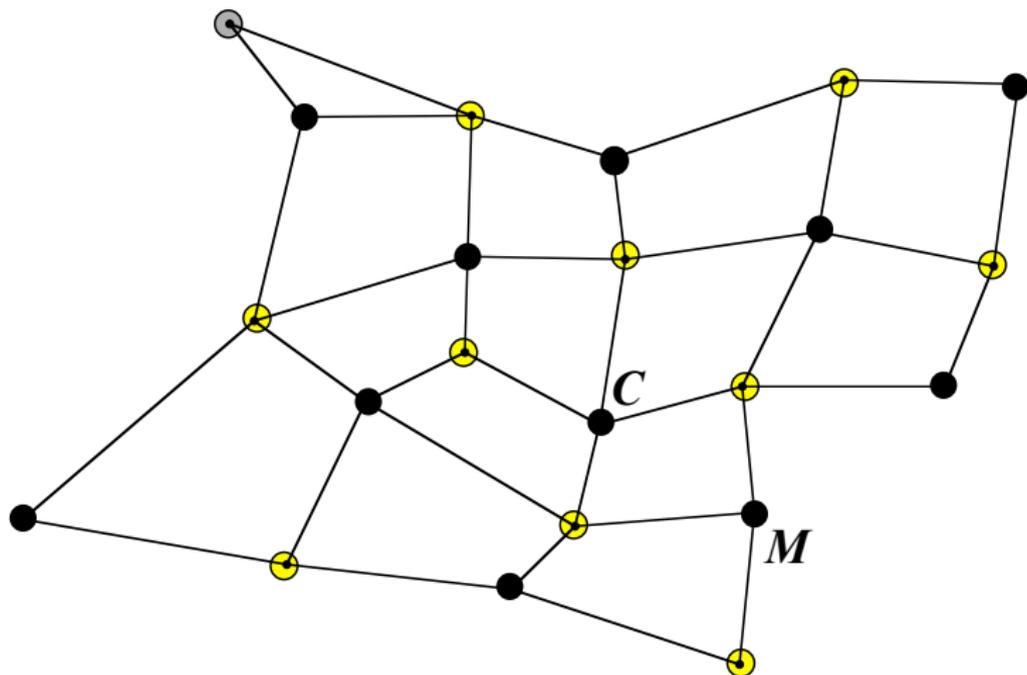
Cat seizes tempo!



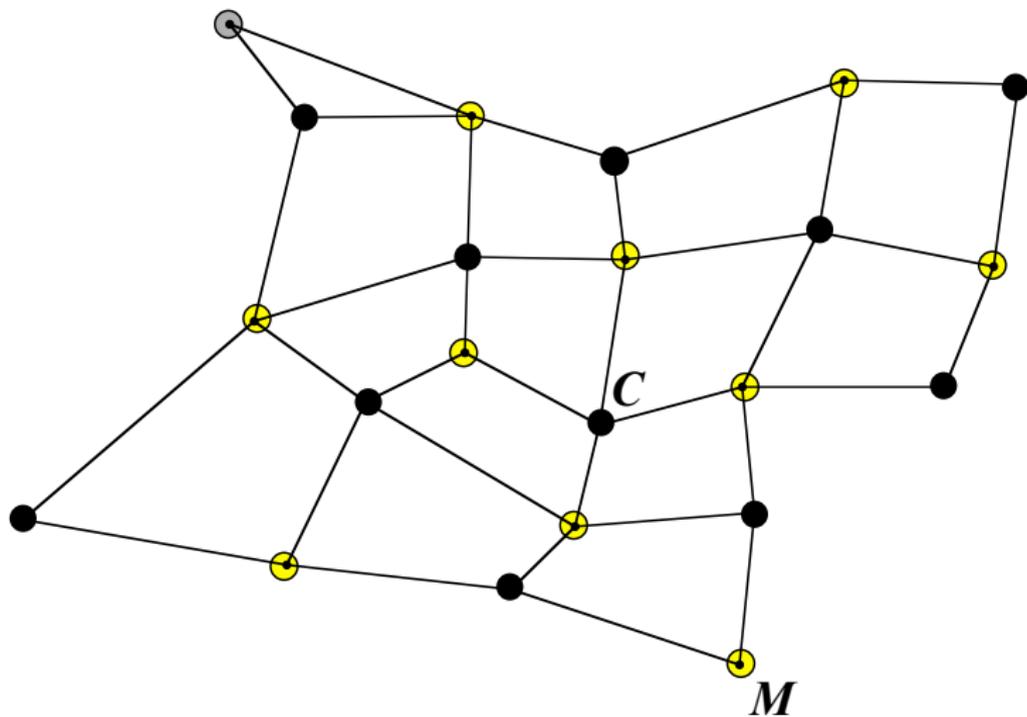
Cat seizes tempo!



Cat seizes tempo!



Cat seizes tempo!



Puppies and Kittens.

We start with, say, 7 kittens and 10 puppies.

Puppies and Kittens.

We start with, say, 7 kittens and 10 puppies.
Two players, A and B, alternate turns.

Puppies and Kittens.

We start with, say, 7 kittens and 10 puppies.

Two players, A and B, alternate turns.

A legal move is removing any number of puppies or any number of kittens or

Puppies and Kittens.

We start with, say, 7 kittens and 10 puppies.

Two players, A and B, alternate turns.

A legal move is removing any number of puppies or any number of kittens or

an equal number of both puppies and kittens.

Puppies and Kittens.

We start with, say, 7 kittens and 10 puppies.

Two players, A and B, alternate turns.

A legal move is removing any number of puppies or any number of kittens or

an equal number of both puppies and kittens.

The winner is the player who makes the last legal move; i.e., the player who removes the last animal.

A Sample Game

Player	kittens	puppies

A Sample Game

Player	kittens	puppies
Start	7	10

A Sample Game

Player	kittens	puppies
Start	7	10
A	5	10

A Sample Game

Player	kittens	puppies
Start	7	10
A	5	10
B	3	8

A Sample Game

Player	kittens	puppies
Start	7	10
A	5	10
B	3	8
A	3	6

A Sample Game

Player	kittens	puppies
Start	7	10
A	5	10
B	3	8
A	3	6
B	3	4

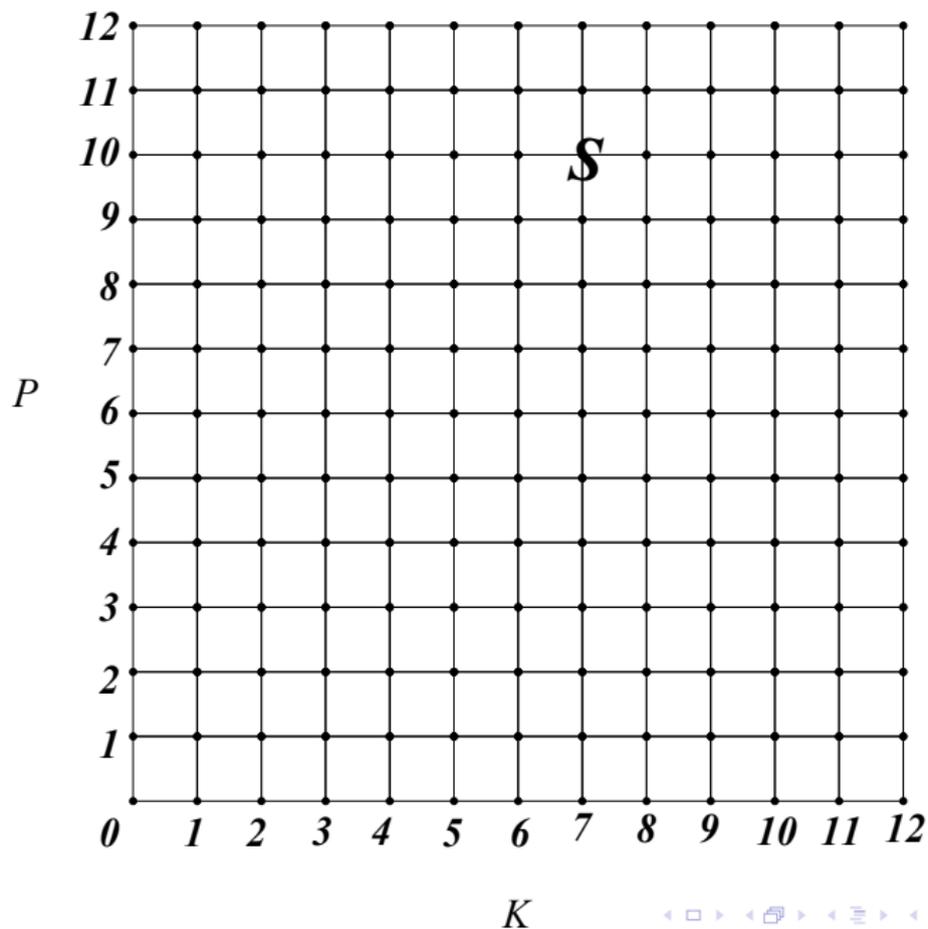
A Sample Game

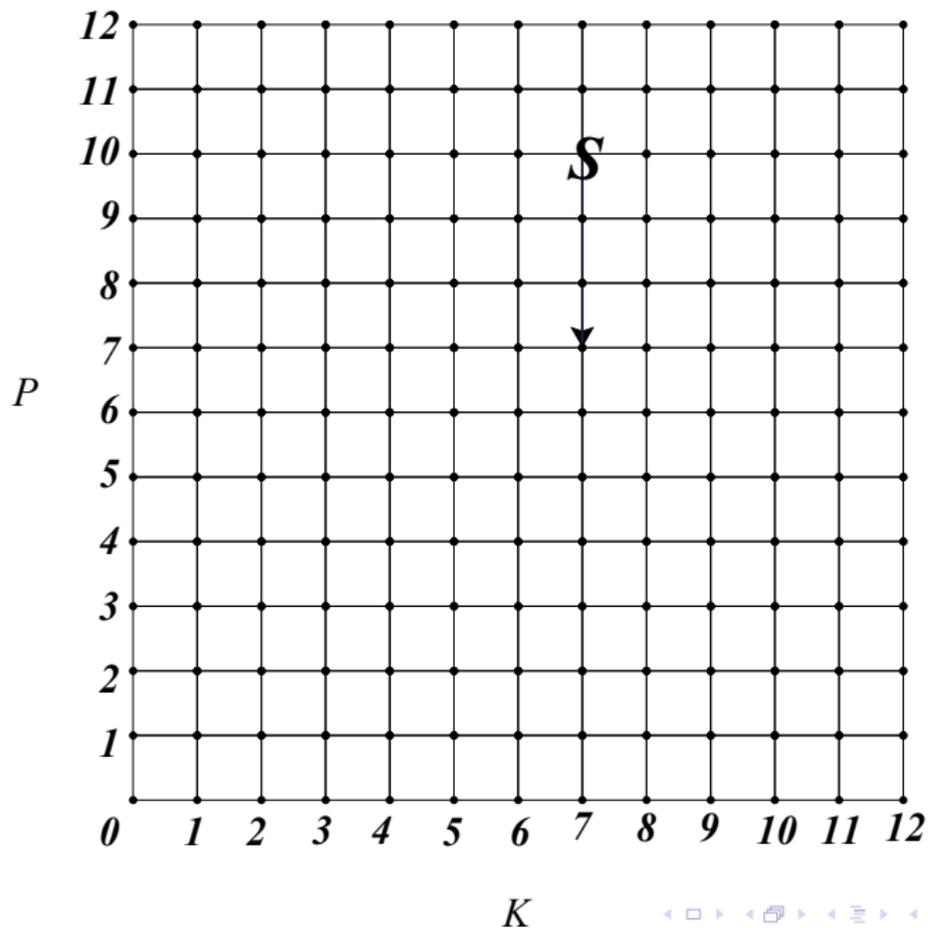
Player	kittens	puppies
Start	7	10
A	5	10
B	3	8
A	3	6
B	3	4
A	1	2

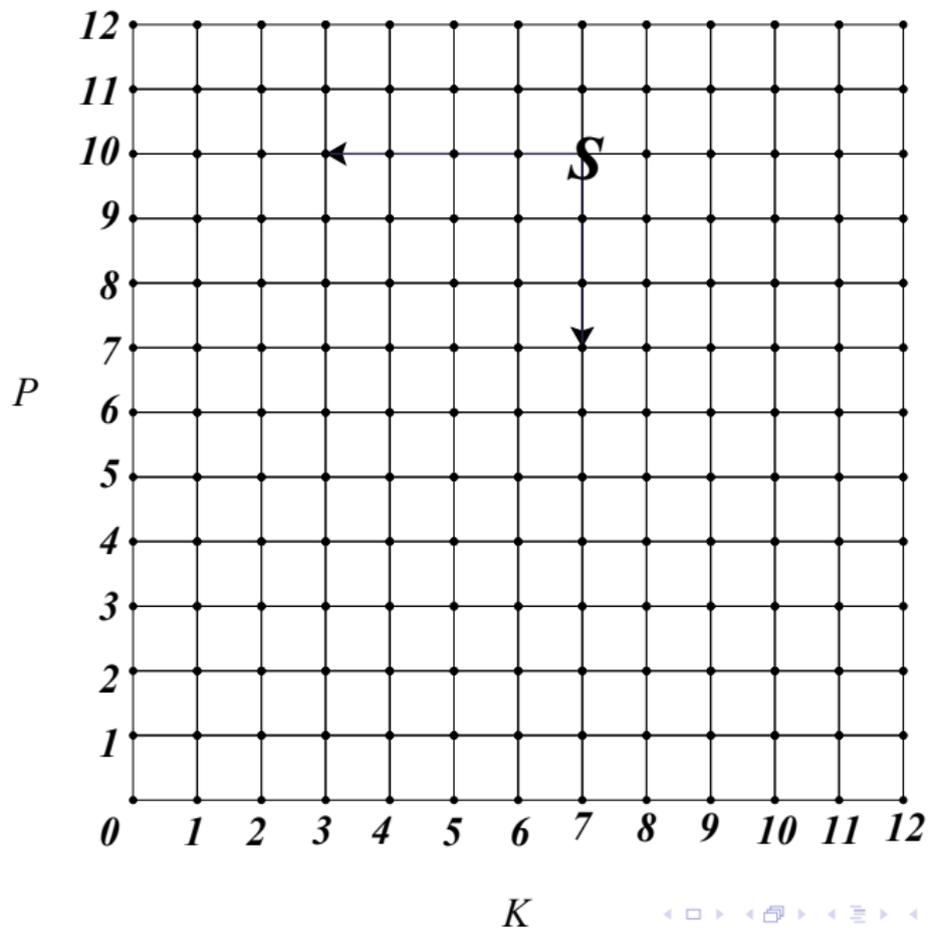
A Sample Game

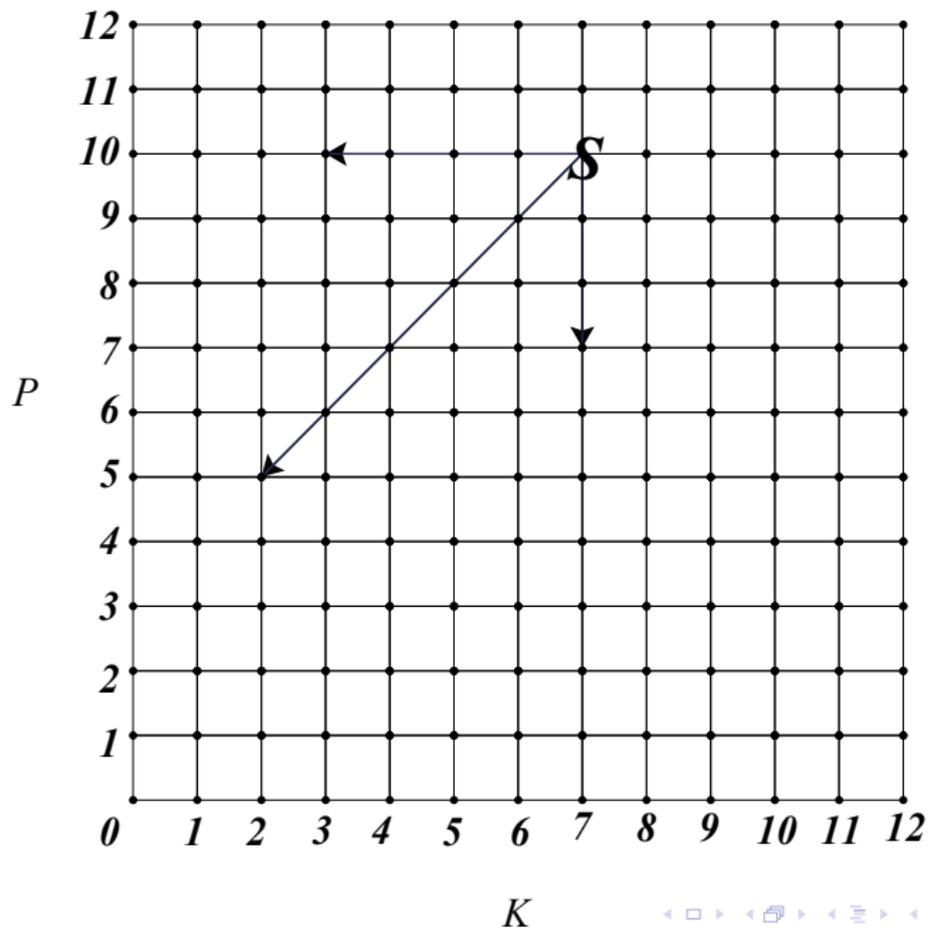
Player	kittens	puppies
Start	7	10
A	5	10
B	3	8
A	3	6
B	3	4
A	1	2

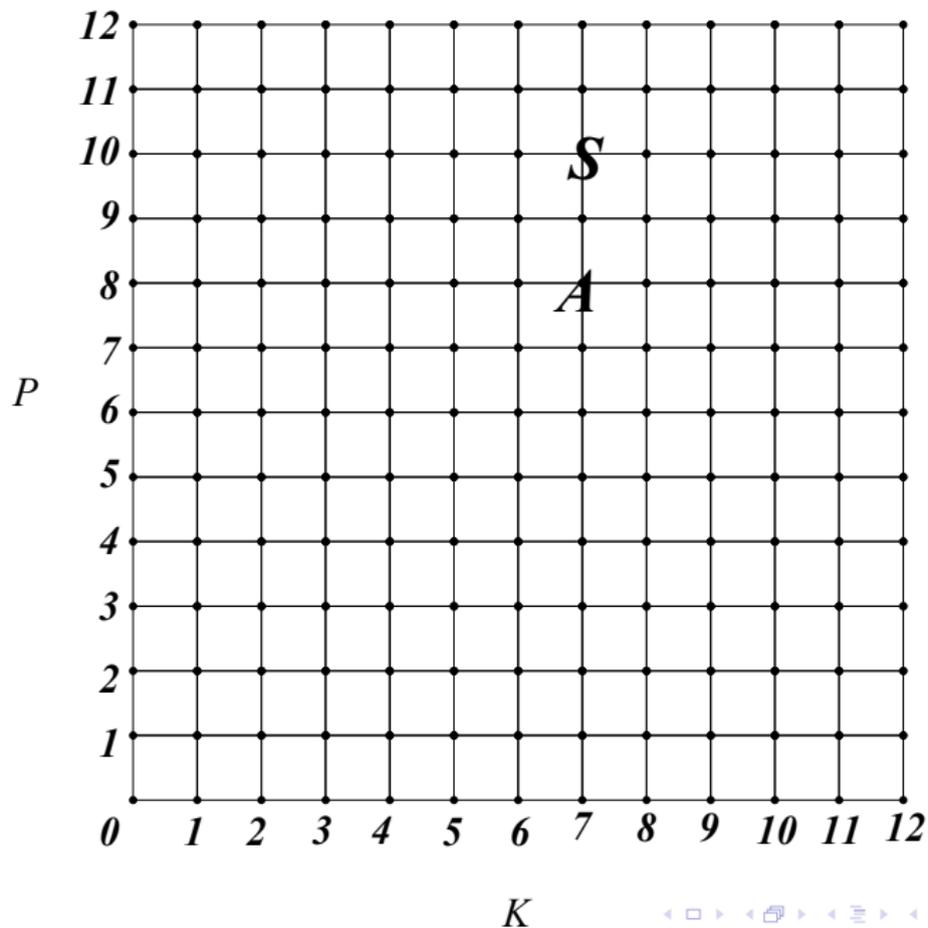
So A wins!

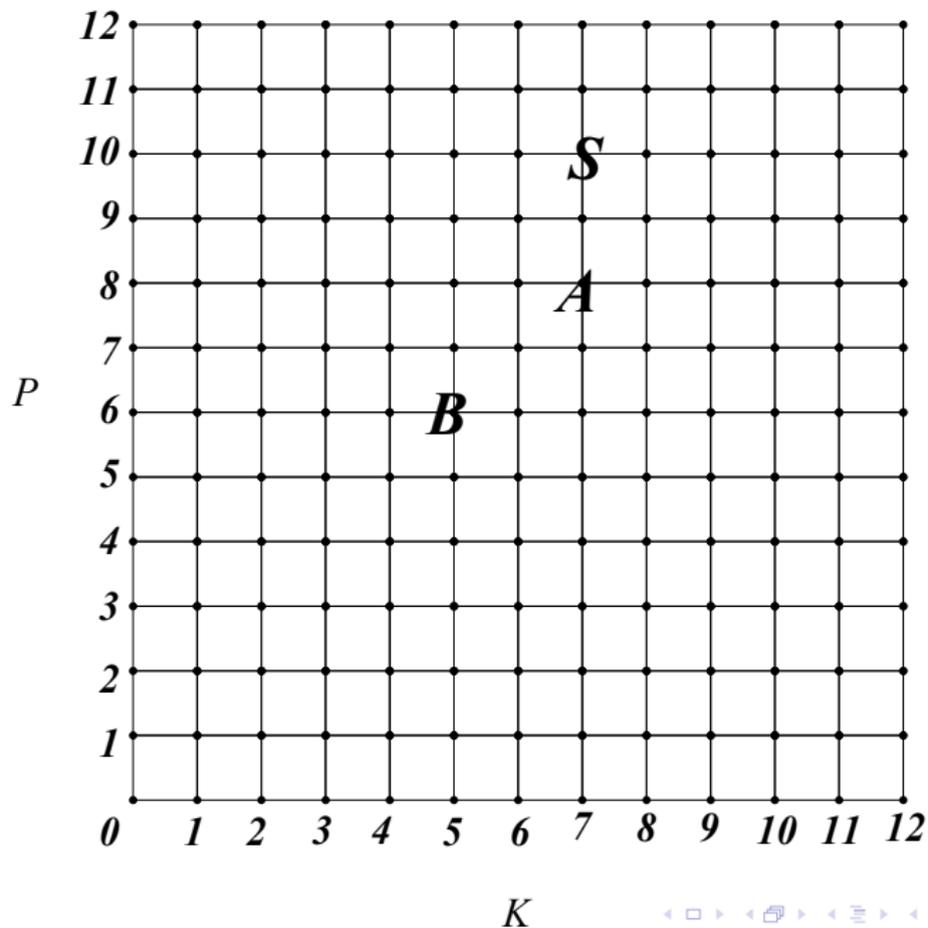


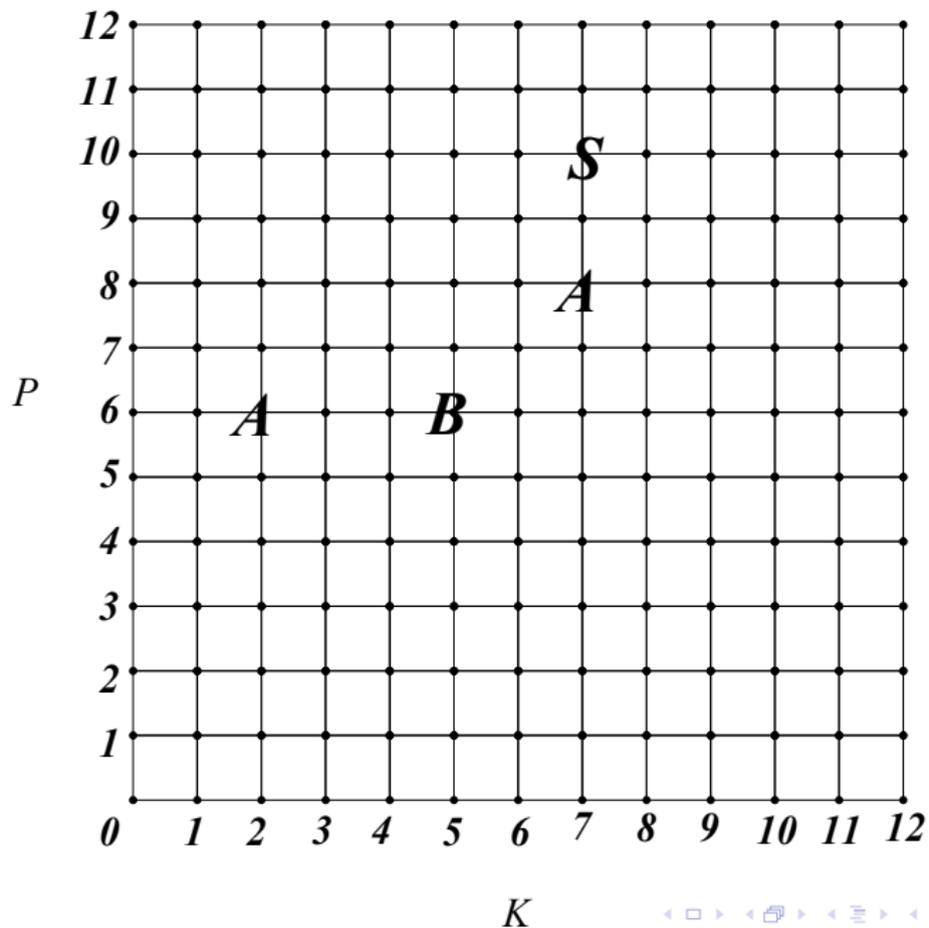


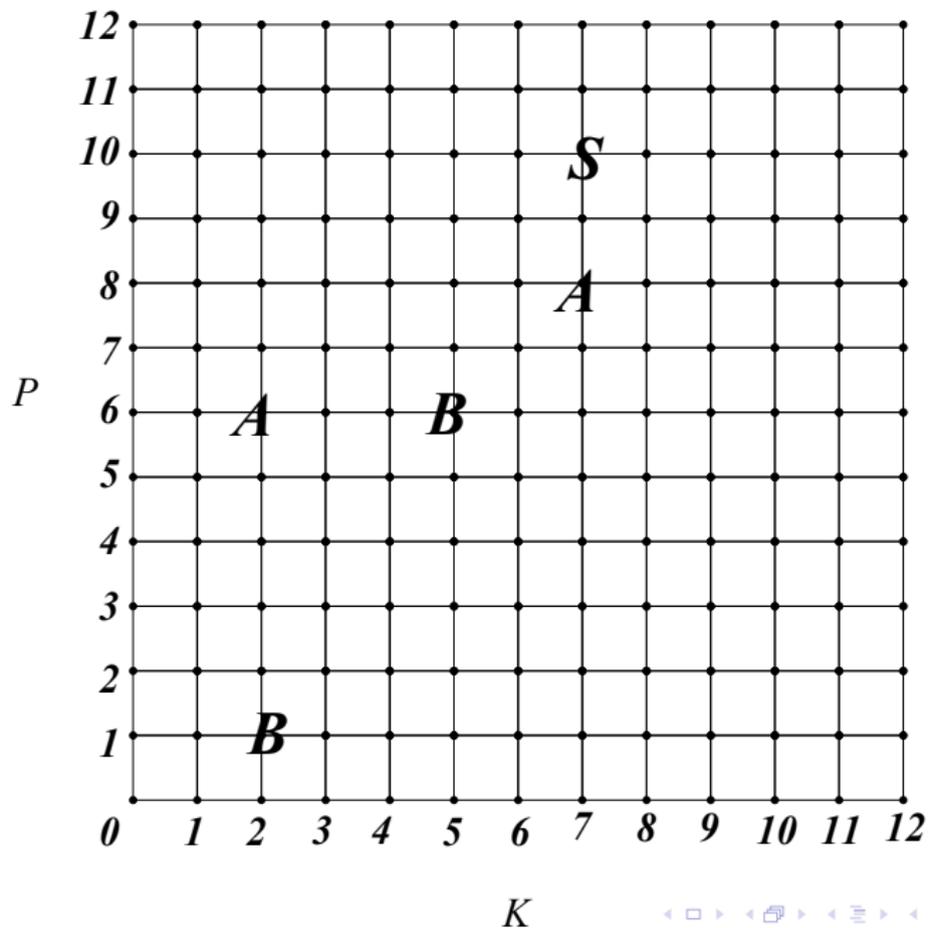
 K

 K









If you present $(2, 1)$ or $(1, 2)$ to your opponent, you've won!

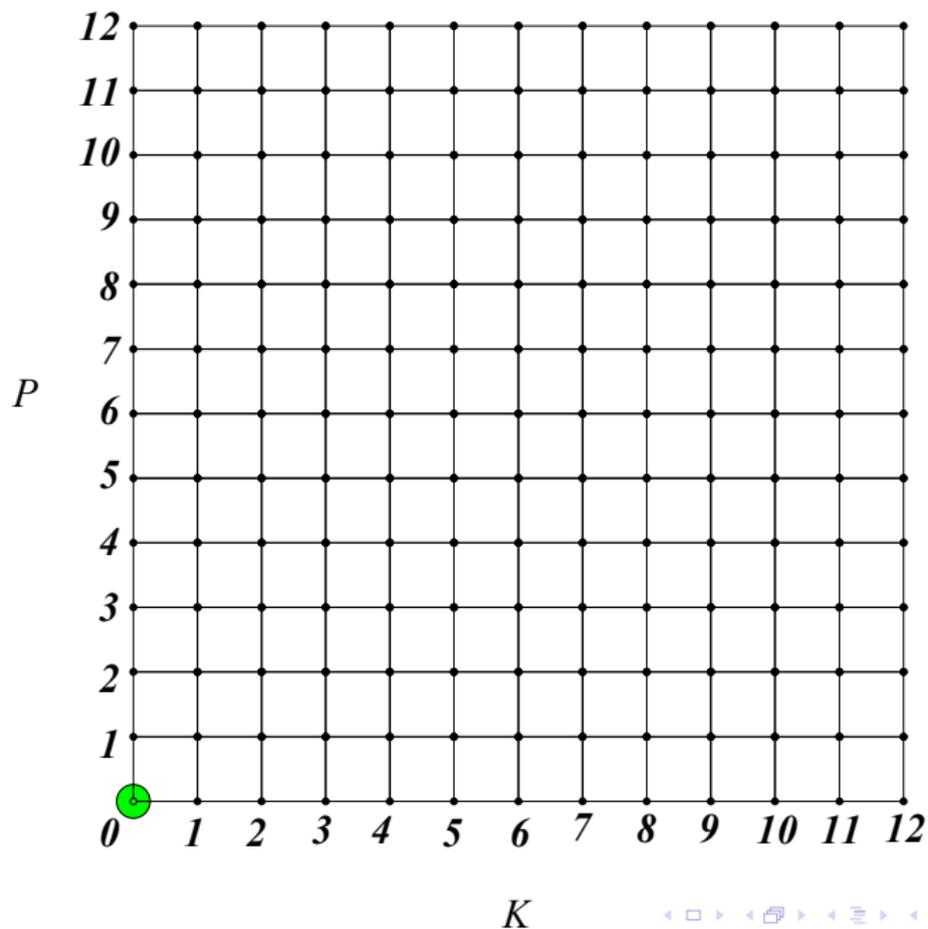
If you present $(2, 1)$ or $(1, 2)$ to your opponent, you've won!

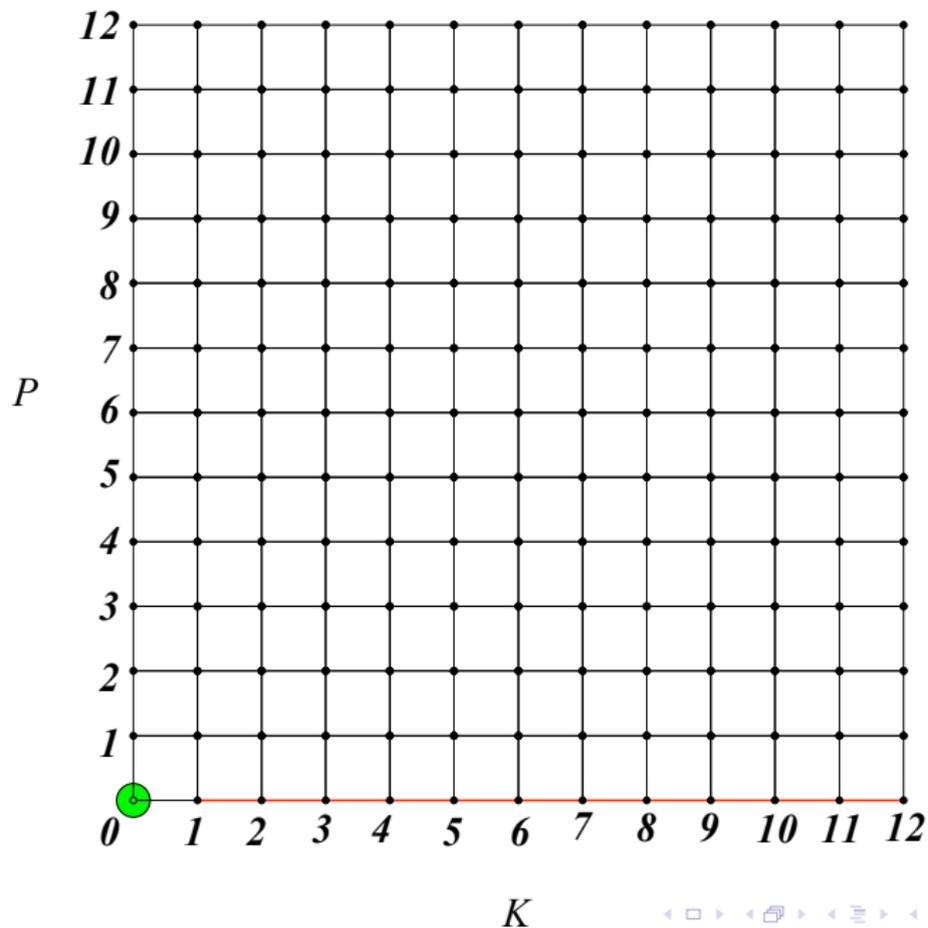
Even simpler: if you present $(0, 0)$ to your opponent, you've won!

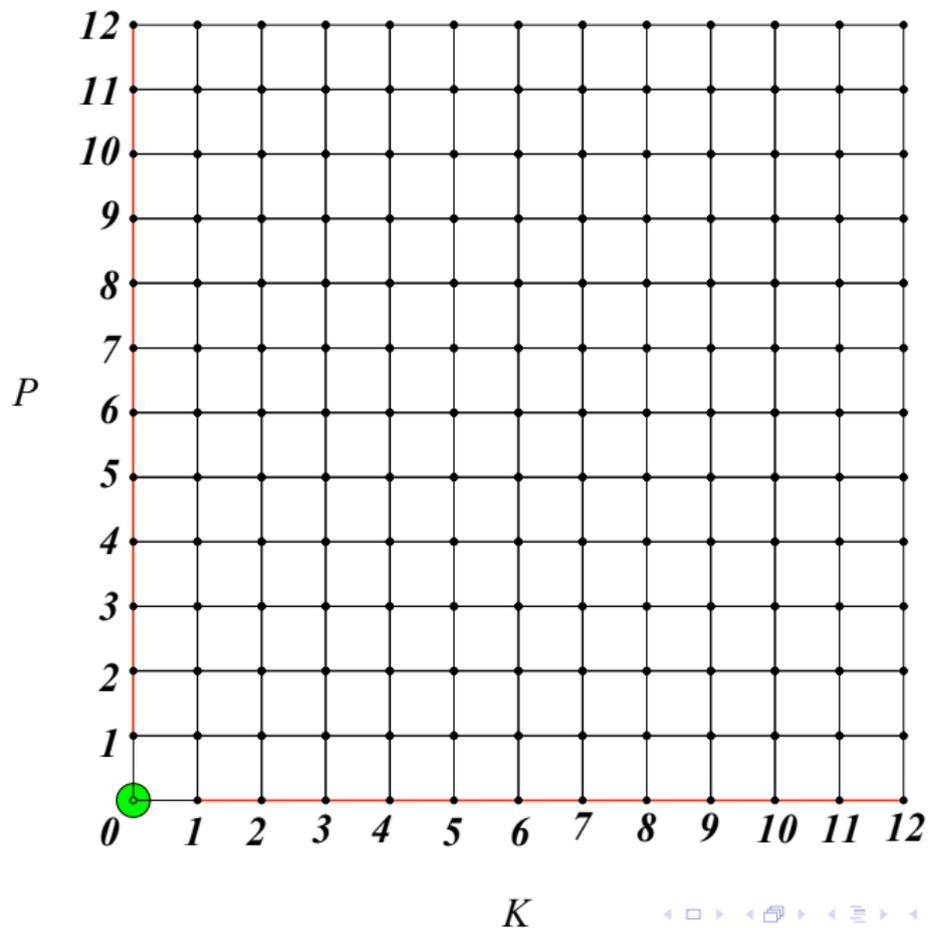
If you present $(2, 1)$ or $(1, 2)$ to your opponent, you've won!

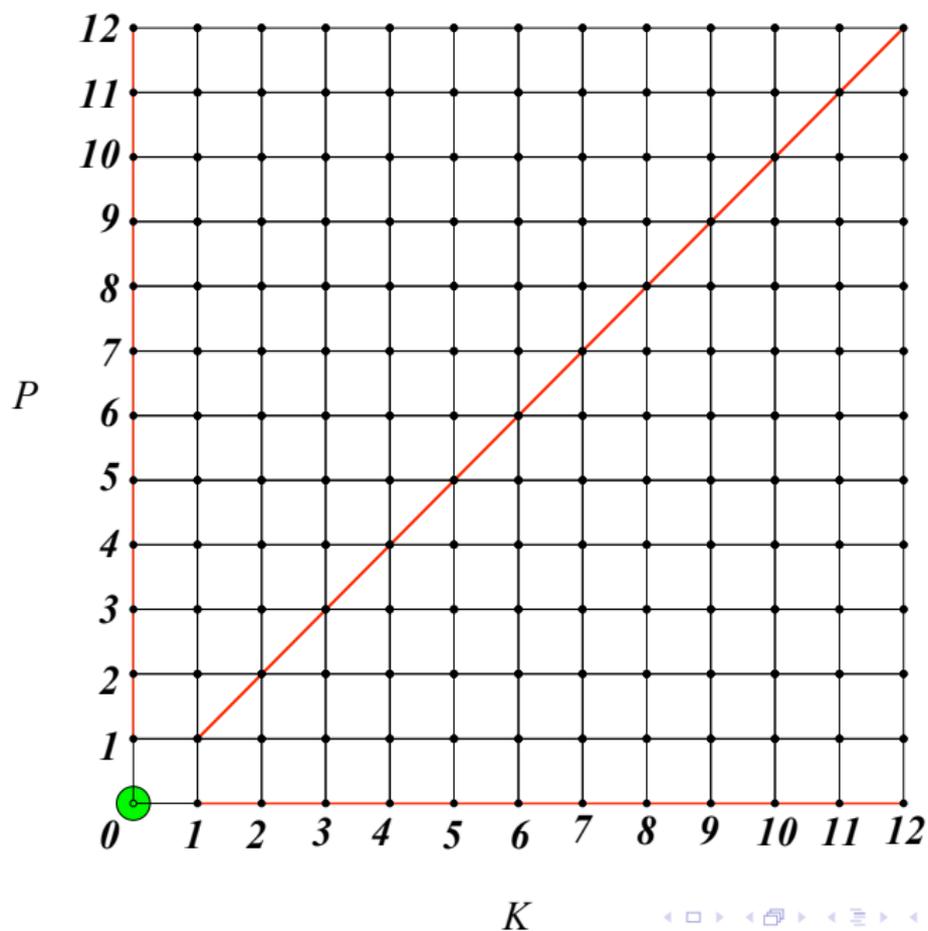
Even simpler: if you present $(0, 0)$ to your opponent, you've won!

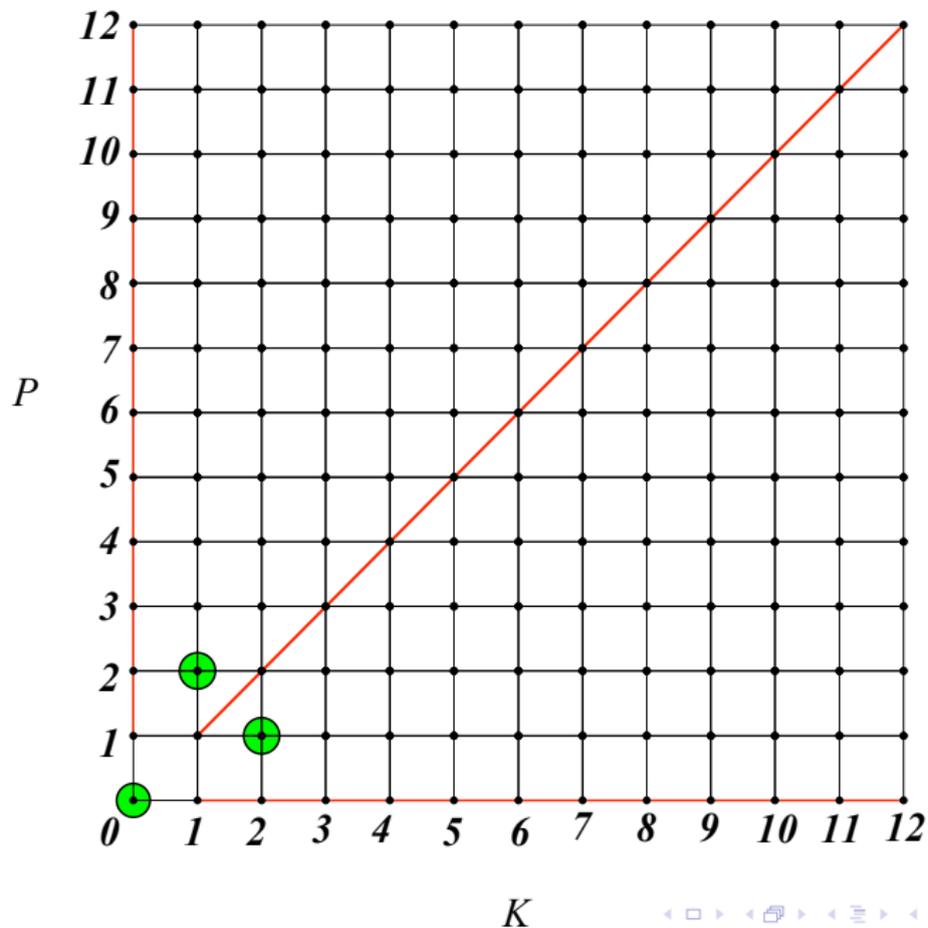
So let's work BACKWARDS from $(0, 0)$.

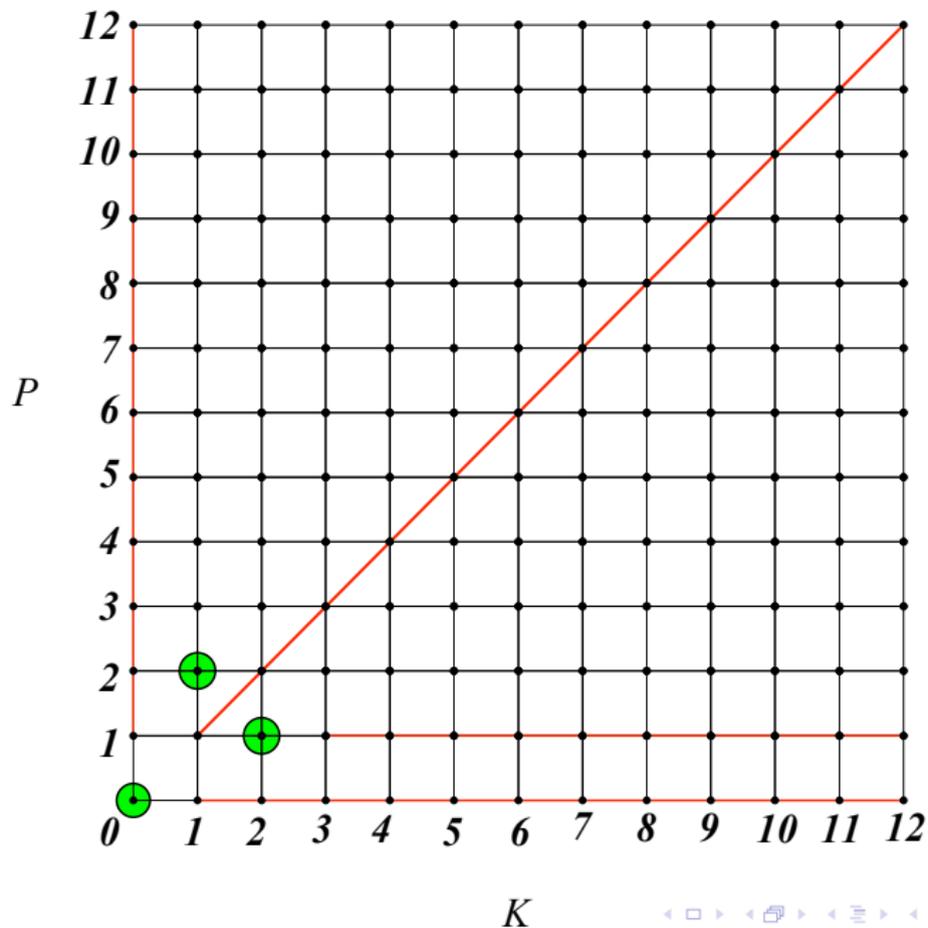
 K

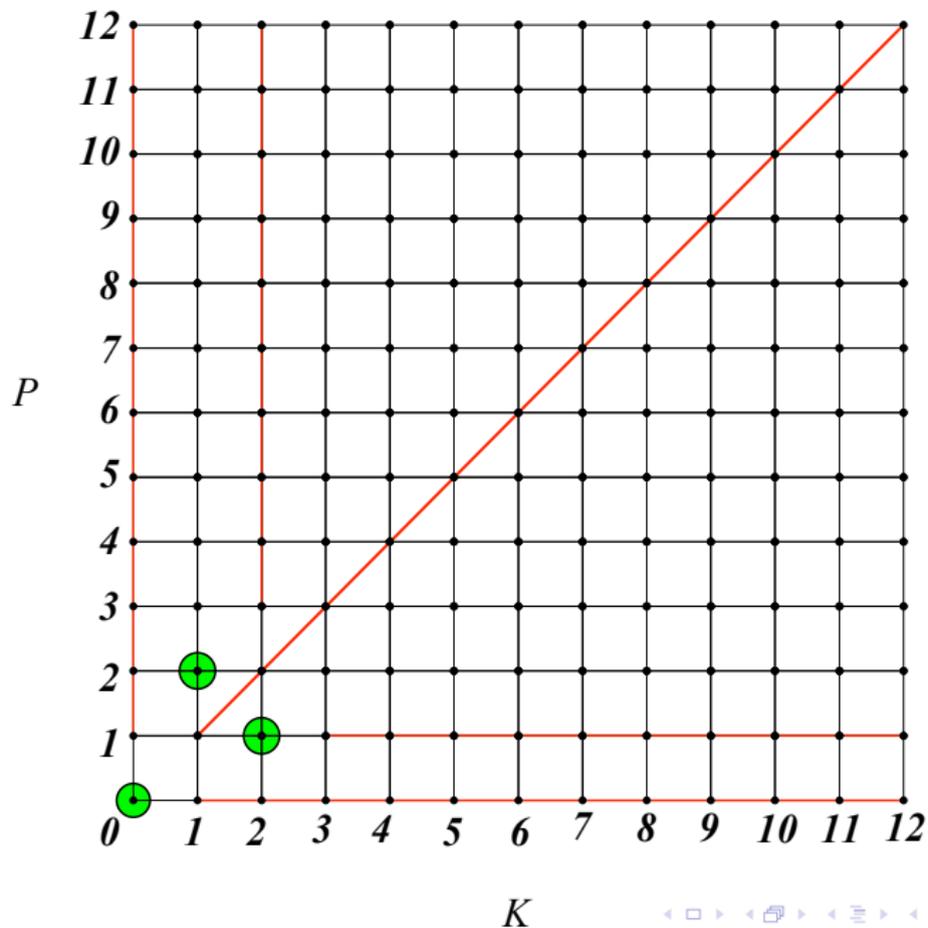


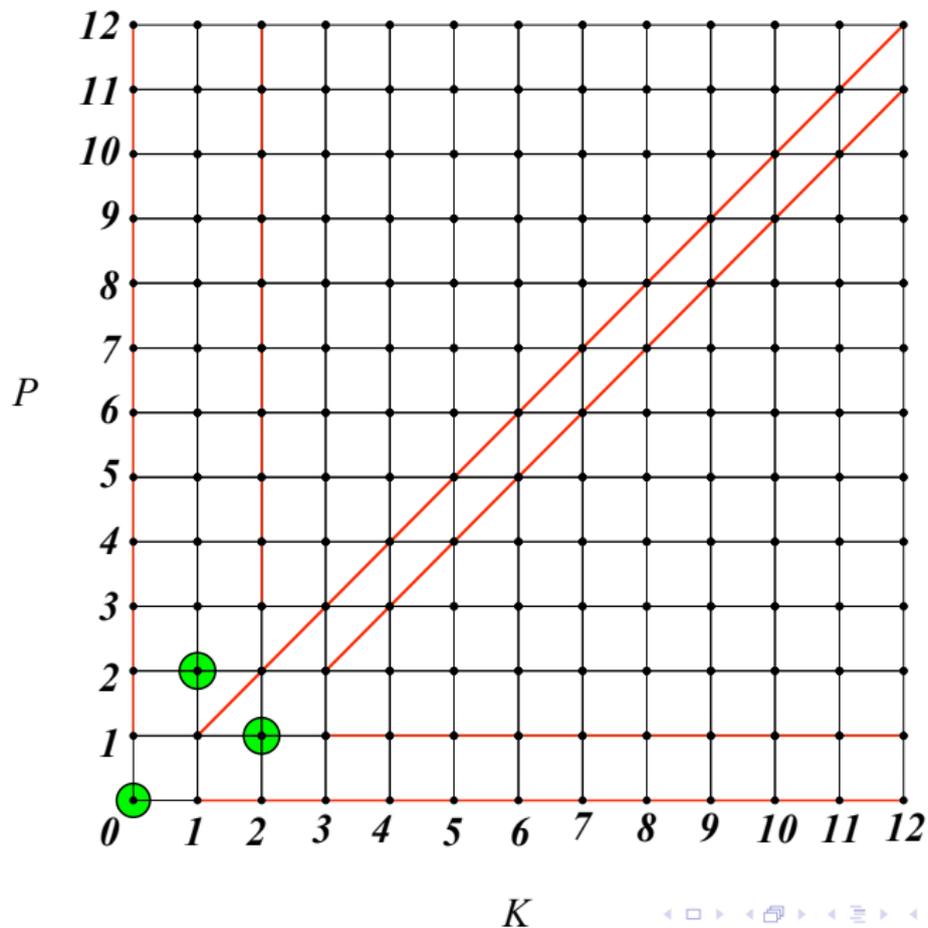
 K

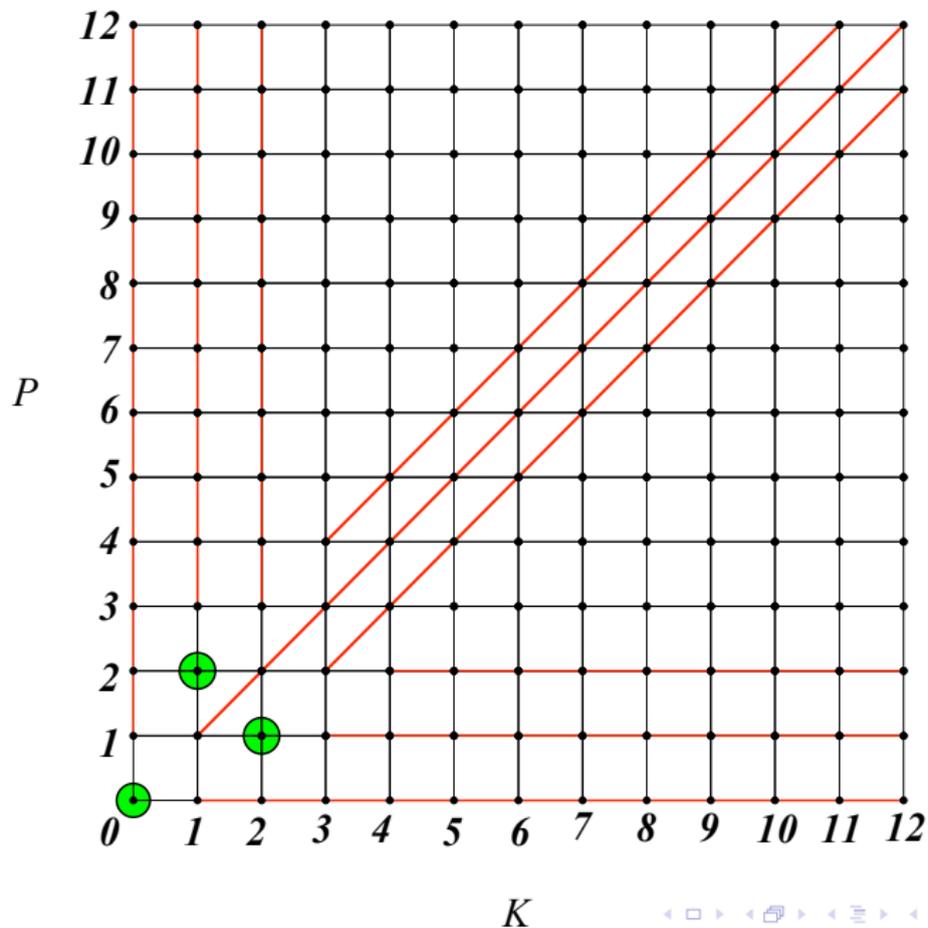
 K

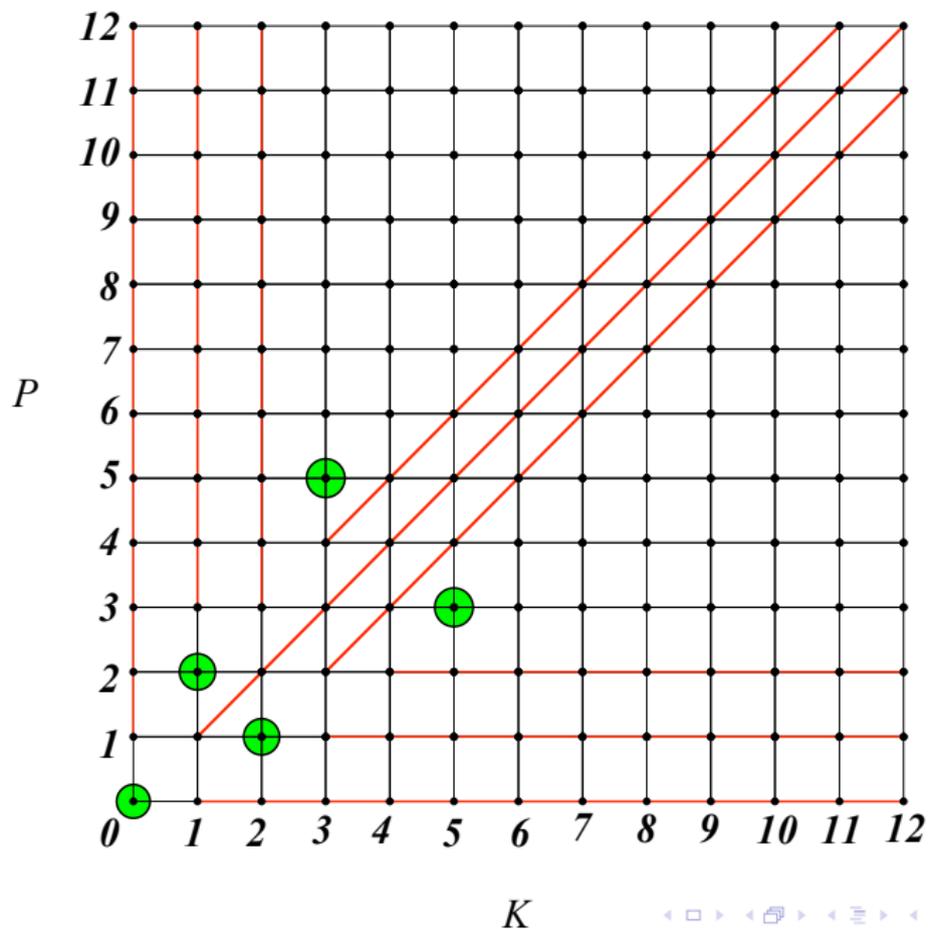
 K

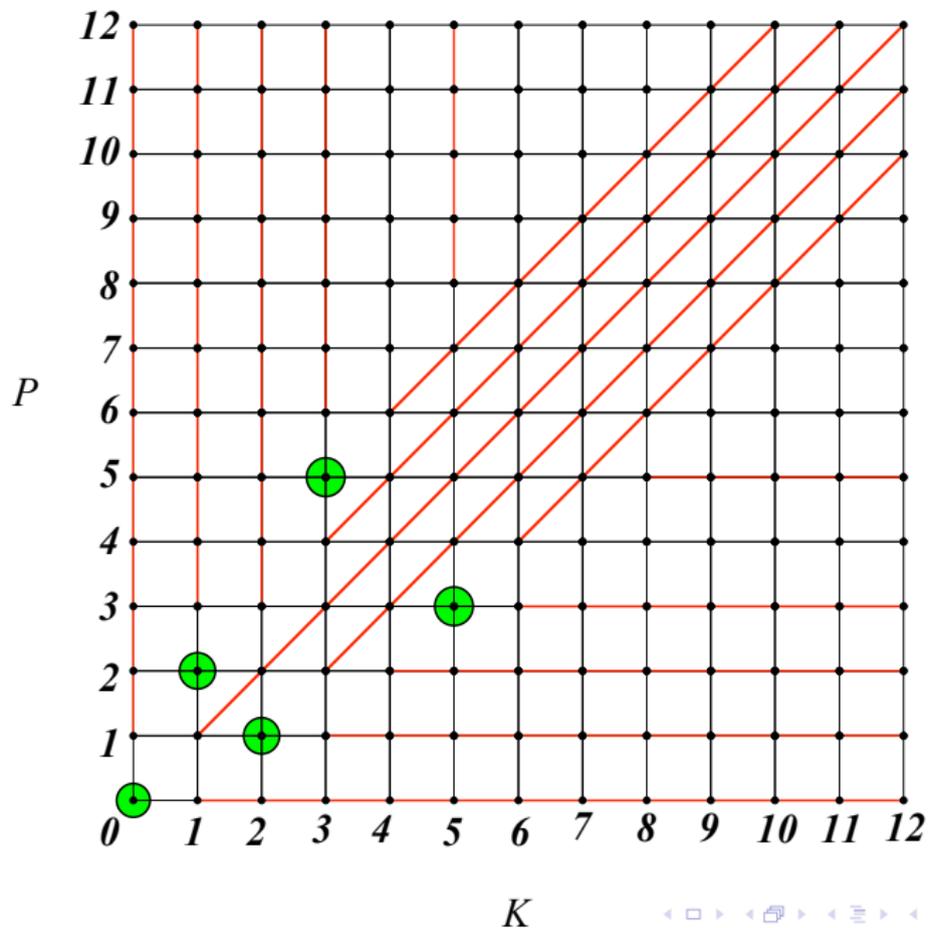
 K

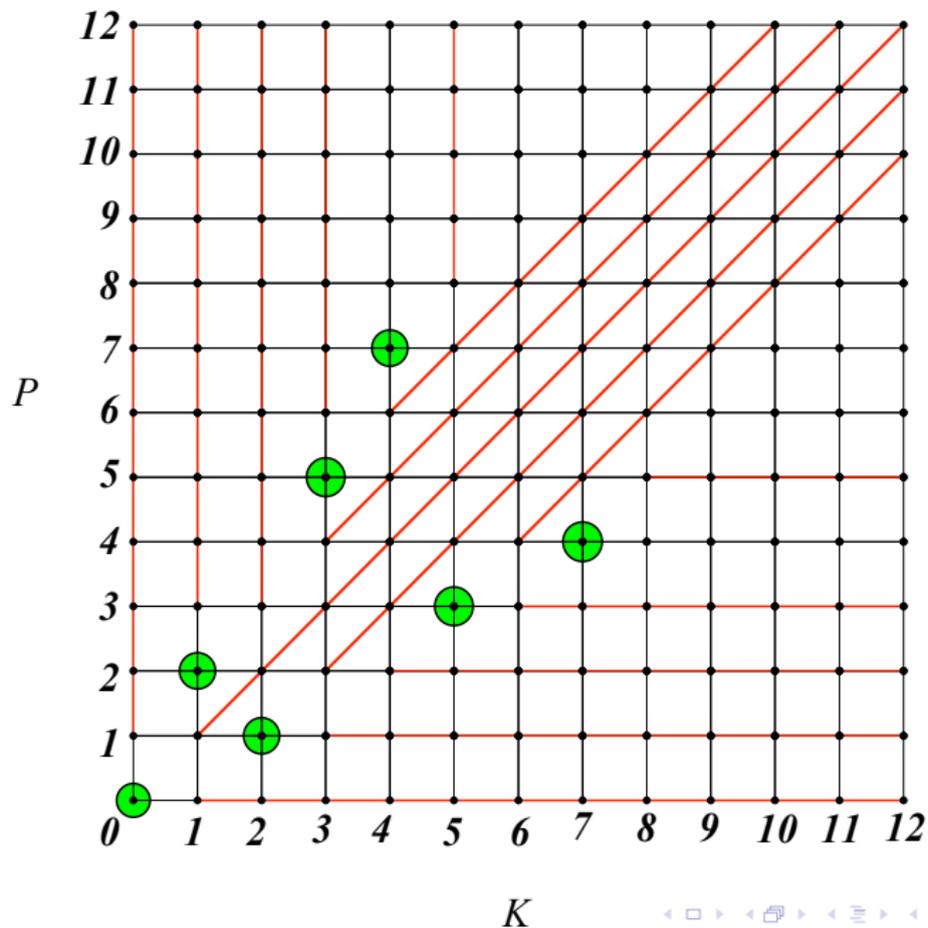
 K

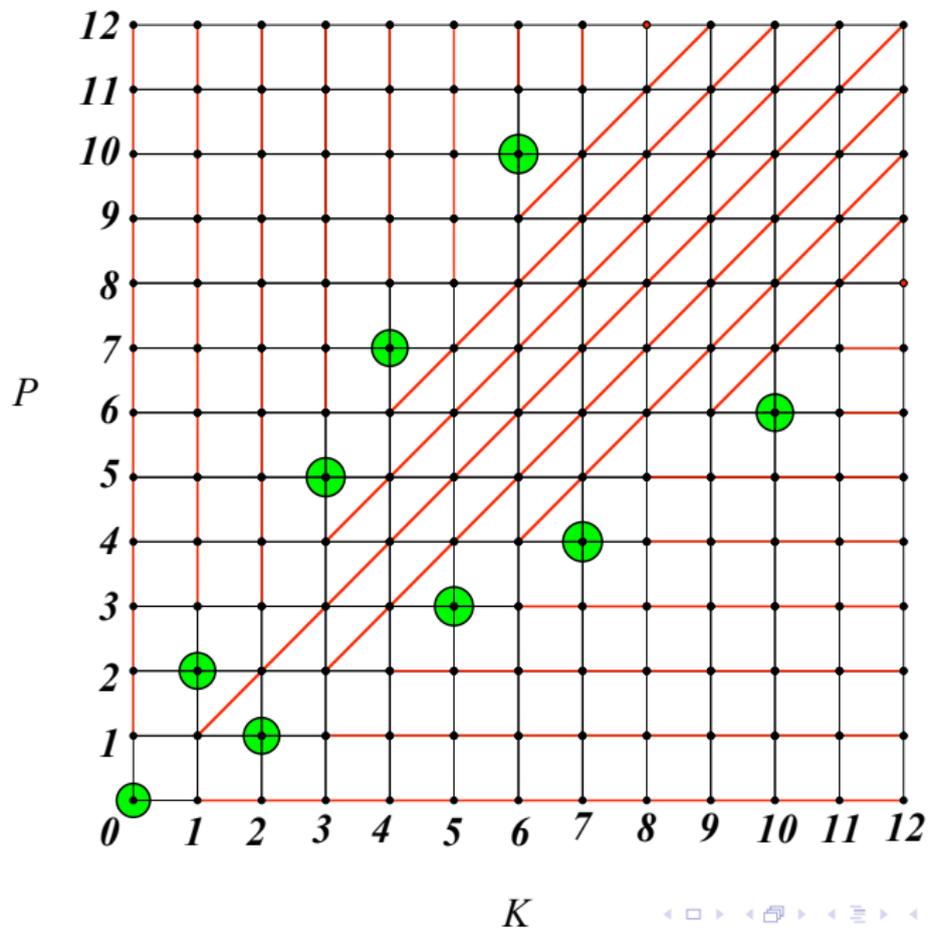
 K

 K

 K





 K

The “Oases” of the Puppies and Kittens game

difference	kittens	puppies

The “Oases” of the Puppies and Kittens game

difference	kittens	puppies
0	0	0

The “Oases” of the Puppies and Kittens game

difference	kittens	puppies
0	0	0
1	1	2

The “Oases” of the Puppies and Kittens game

difference	kittens	puppies
0	0	0
1	1	2
2	3	5

The “Oases” of the Puppies and Kittens game

difference	kittens	puppies
0	0	0
1	1	2
2	3	5
3	4	7

The “Oases” of the Puppies and Kittens game

difference	kittens	puppies
0	0	0
1	1	2
2	3	5
3	4	7
4	6	10

The “Oases” of the Puppies and Kittens game

difference	kittens	puppies
0	0	0
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13

The “Oases” of the Puppies and Kittens game

In fact, there is an explicit formula for the oasis.

The “Oases” of the Puppies and Kittens game

In fact, there is an explicit formula for the oasis.
Define the “floor function” $\lfloor x \rfloor$ to be the greatest integer less than or equal to x .

The “Oases” of the Puppies and Kittens game

In fact, there is an explicit formula for the oasis.
Define the “floor function” $\lfloor x \rfloor$ to be the greatest integer less than or equal to x . For example, $\lfloor \pi \rfloor = 3$.

The “Oases” of the Puppies and Kittens game

Then if the difference is n , the oasis is

$$(\lfloor n\phi \rfloor, \lfloor n\phi \rfloor + n)$$

The “Oases” of the Puppies and Kittens game

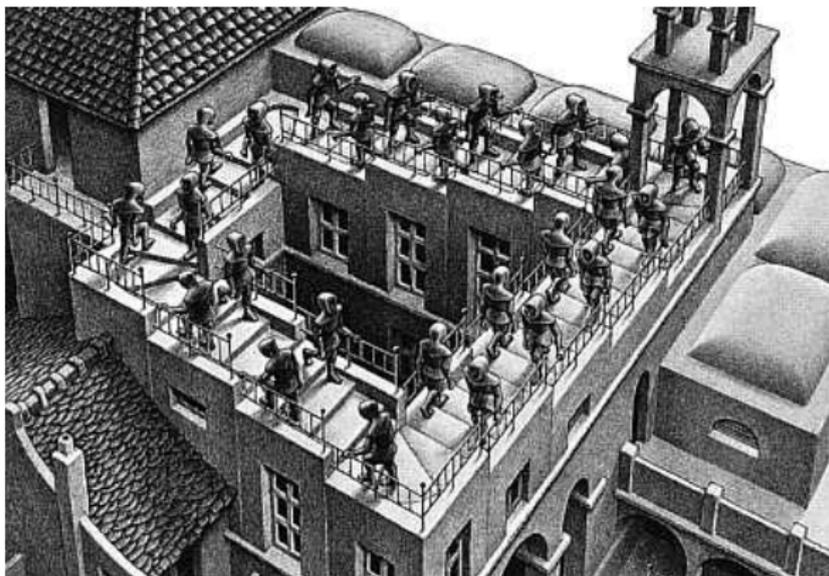
Then if the difference is n , the oasis is

$$(\lfloor n\phi \rfloor, \lfloor n\phi \rfloor + n)$$

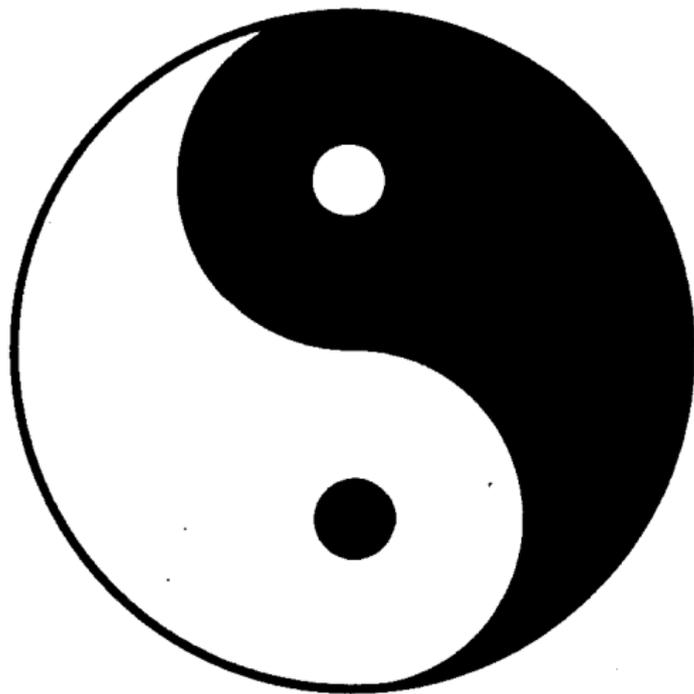
where ϕ is the *Golden Ratio*:

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.61803.$$

The Morals of the Stories



The Morals of the Stories



Questions? Contact me at zeitzp@usfca.edu