The Mathemagic of Magic Squares

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April 15, 2012

Warm-Up

The 15 Game

Players take turns choosing numbers between 1 and 9, without repeats. The first player to choose 3 numbers that add up to 15 wins.

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1 2 3 4 5 6 7 8 9

Player 1: Player 2:

Warm-Up

The 15 Game

Players take turns choosing numbers between 1 and 9, without repeats. The first player to choose 3 numbers that add up to 15 wins.

1 2 4 5 6 7 8 9

Player 1:
3

Player 2:

Warm-Up

The 15 Game

Players take turns choosing numbers between 1 and 9, without repeats. The first player to choose 3 numbers that add up to 15 wins.

1 4 5 6 7 8 9

Player 1:
3

Player 2:
2

Warm-Up

The 15 Game

Players take turns choosing numbers between 1 and 9, without repeats. The first player to choose 3 numbers that add up to 15 wins.

1 4 5 7 8 9

Player 1:
3, 6

Player 2:
2

Warm-Up

The 15 Game

Players take turns choosing numbers between 1 and 9, without repeats. The first player to choose 3 numbers that add up to 15 wins.

1 4 7 8 9

Player 1:
3, 6

Player 2:
2, 5

Warm-Up

The 15 Game

Players take turns choosing numbers between 1 and 9, without repeats. The first player to choose 3 numbers that add up to 15 wins.

1 4 7 9

Player 1:
3, 6, 8

Player 2:
2, 5

Warm-Up

The 15 Game

Players take turns choosing numbers between 1 and 9, without repeats. The first player to choose 3 numbers that add up to 15 wins.

4 7 9

Player 1:
3, 6, 8

Player 2:
2, 5, 1

Warm-Up

The 15 Game

Players take turns choosing numbers between 1 and 9, without repeats. The first player to choose 3 numbers that add up to 15 wins.

7 9

Player 1:
3, 6, 8, 4

Player 2:
2, 5, 1

Warm-Up

The 15 Game

Players take turns choosing numbers between 1 and 9, without repeats. The first player to choose 3 numbers that add up to 15 wins.

7 9

Player 1:
3, 6, 8, 4

Player 2:
2, 5, 1

Outline

Definition

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A **magic square** is a filling of an $n \times n$ square with the numbers $1, 2, \ldots, n^2$ so that the rows, columns, and diagonals all sum to the same number.

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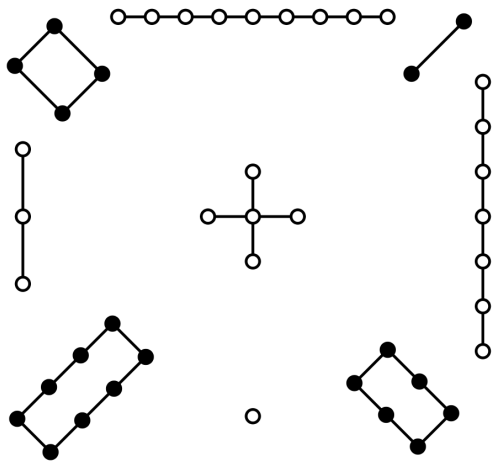
1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

The Lo Shu Square

Lo Shu Square: $\sim$ 650 BCE



Magic Sum 15 is the number of days in the 24 cycles of the Chinese solar year.

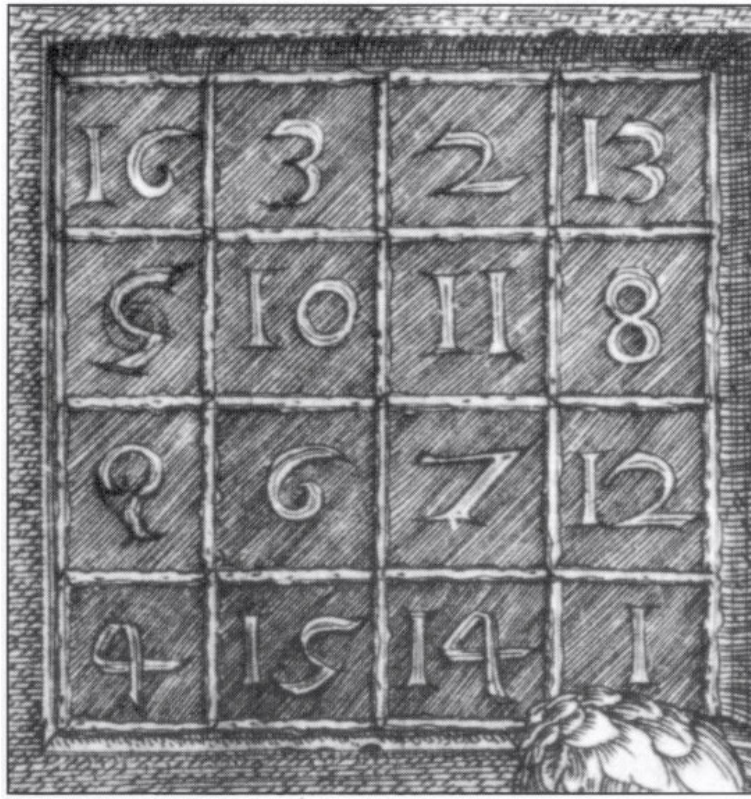
The Chautisa Yantra

Chautisa Yantra: Parshvanath Jain temple in Khajuraho, India (10th century)

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

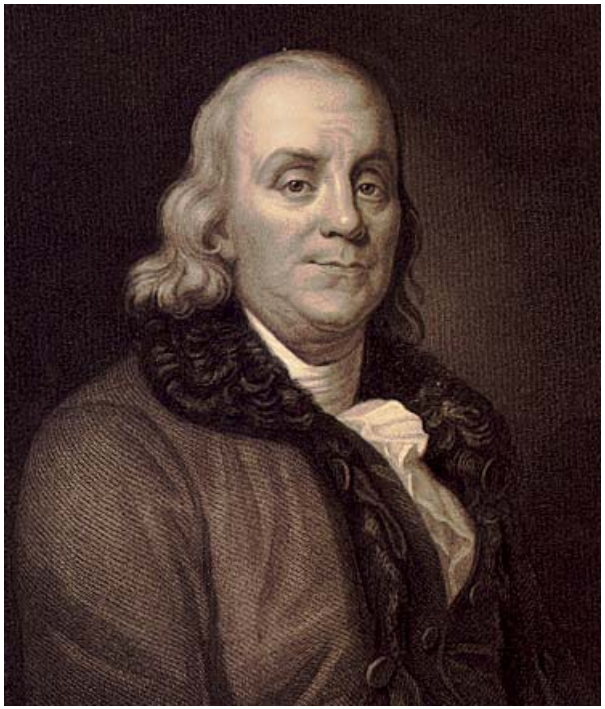
Dürer's Square

Albrecht Dürer: *Melencolia I* (1514)



Benjamin Franklin's Squares

"The Governor put me into the commission of the Peace; the Corporation of the City chose me of the Common Council, and soon after an Alderman; and the Citizens at large chose me a Burgess to represent them in Assembly.
This latter Station was the more agreeable to me, as I was at length tired with sitting there to hear Debates in which as Clerk I could take no part, and which were often so unentertaining, that I was induced to amuse myself with making magic squares, or circles, or anything to avoid weariness."



Benjamin Franklin's Magic Square

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

The Magic Sum

Question: What is the magic sum for an $n \times n$ magic square?

?	?	?	$\cdots$	?	S
?	?	?	$\cdots$	?	S
?	?	?	$\cdots$	?	S
?	?	?	$\cdots$	?	$\vdots$
?	?	?	$\cdots$	?	S
					$n \cdot S$

So

$$n \cdot S = 1 + 2 + 3 + \cdots + n^2$$

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?	?	?	$\cdots$	?	S
?	?	?	$\cdots$	?	$\vdots$
?	?	?	$\cdots$	?	S
					$n \cdot S$

So

$$\begin{aligned} n \cdot S &= 1 + 2 + 3 + \cdots + n^2 \\ &= \frac{n^2(n^2+1)}{2} \end{aligned}$$

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Question: What is the magic sum for an $n \times n$ magic square?

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?	?	?	$\cdots$	?	S
?	?	?	$\cdots$	?	S
?	?	?	$\cdots$	?	$\vdots$
?	?	?	$\cdots$	?	S
					$n \cdot S$

So

$$\begin{aligned} n \cdot S &= 1 + 2 + 3 + \cdots + n^2 \\ &= \frac{n^2(n^2+1)}{2} \\ S &= \frac{n(n^2+1)}{2} \end{aligned}$$

The Magic Sum

The Magic Sum

The magic sum for an $n \times n$ magic square is

$$\frac{n(n^2+1)}{2}.$$

Example:

$$n = 3: \quad S = \frac{3 \cdot (3^2+1)}{2} = \frac{3 \cdot 10}{2} = 15$$

$$n = 4: \quad S = \frac{4 \cdot (4^2+1)}{2} = \frac{4 \cdot 17}{2} = 34$$

$$n = 5: \quad S = \frac{5 \cdot (5^2+1)}{2} = \frac{5 \cdot 26}{2} = 65$$

$$n = 8: \quad S = \frac{8 \cdot (8^2+1)}{2} = \frac{8 \cdot 65}{2} = 260$$

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Rules

Two players take turns choosing numbers between 1 and 9. The objective is to collect three numbers that sum to 15.

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Winning collections:

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Winning collections:

$$1+9+5$$
$$1+8+6$$
$$2+9+4$$
$$2+8+5$$
$$2+7+6$$
$$3+8+4$$
$$3+7+5$$
$$4+6+5$$

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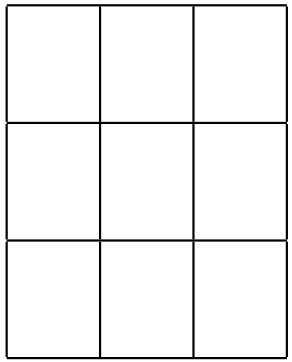

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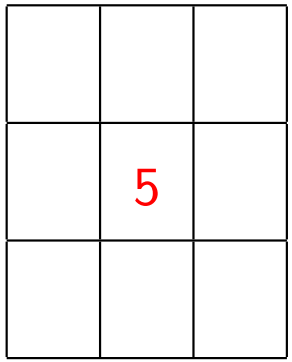

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8	1	6
	5	

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3	5	7
4	9	2

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Rules

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Winning collections:

Player 1:

Player 2:

8	1	6
3	5	7
4	9	2

The 15 game

Rules

Two players take turns choosing numbers between 1 and 9. The objective is to collect three numbers that sum to 15.

Winning collections:

Player 1:

3

Player 2:

8	1	6
3	5	7
4	9	2

The 15 game

Rules

Two players take turns choosing numbers between 1 and 9. The objective is to collect three numbers that sum to 15.

Winning collections:

Player 1:

3

Player 2:

8	1	6
X	5	7
4	9	2

The 15 game

Rules

Two players take turns choosing numbers between 1 and 9. The objective is to collect three numbers that sum to 15.

Winning collections:

Player 1:

3

Player 2:

2

8	1	6
X	5	7
4	9	2

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Two players take turns choosing numbers between 1 and 9. The objective is to collect three numbers that sum to 15.

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Player 1:

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Player 2:

2

8	1	6
X	5	7
4	9	O

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3, 6

Player 2:

2

8	1	6
X	5	7
4	9	O

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3, 6

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2

8	1	X
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Winning collections:

Player 1:

3, 6, 8

Player 2:

2, 5

X	1	X
X	O	7
4	9	O

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Player 2:

2, 5, 1

X	O	X
X	O	7
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Rules

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Winning collections:

Player 1:

3, 6, 8, 4

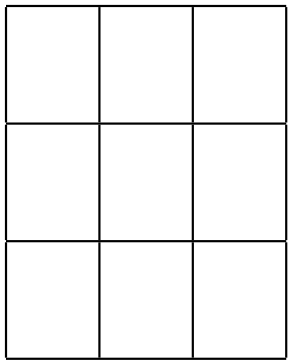
Player 2:

2, 5, 1

X	O	X
X	O	7
X	9	O

Constructing Odd Magic Squares

1. Place 1 in the middle of the top row.
2. Having placed number i, place number $i + 1$:
   1. One square to the northeast of i, if you can (wrapping if necessary).
   2. One square to the south of i, otherwise.



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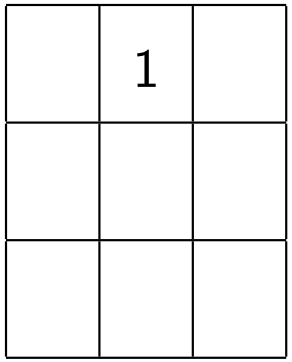

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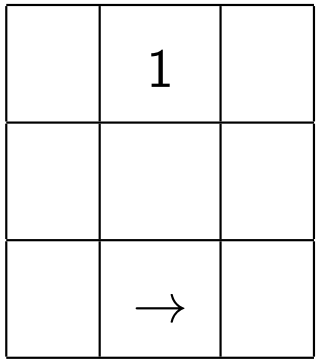

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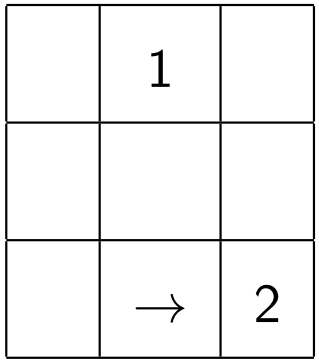

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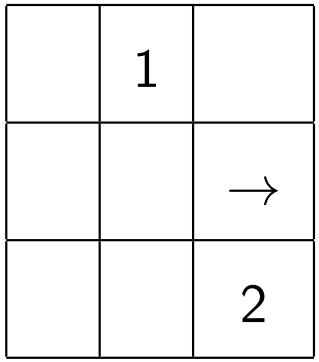

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	1	
3		$\rightarrow$
		2

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3		
4		2

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	1	
3	5	
4		2

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	1	6
3	5	
4		2

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	1	6
3	5	7
4		2

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8	1	6
3	5	7
4		2

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17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

What about even Magic Squares?

When $n = 2 \cdot (2m + 1)$

1. Start with a $2m + 1 \times 2m + 1$ magic square.
2. Fill another $2m + 1 \times 2m + 1$ square with the letters L, U, and X as follows:

What about even Magic Squares?

When $n = 2 \cdot (2m + 1)$

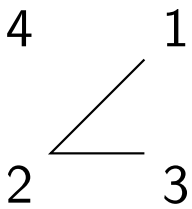
1. Start with a $2m + 1 \times 2m + 1$ magic square.
2. Fill another $2m + 1 \times 2m + 1$ square with the letters L, U, and X as follows:
   1. Fill the first $m + 1$ rows with L.
   2. Fill the next row with U.
   3. Fill the remaining rows with X.
   4. Replace the middle entry of the U row with the L above it.

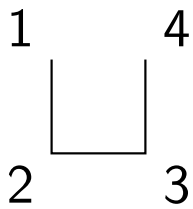
8	1	6
3	5	7
4	9	2

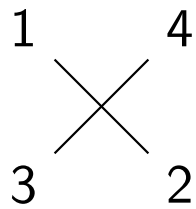
L	L	L
L	U	L
U	L	U

The LUX Method

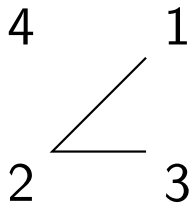
3. Replace each square in the LUX grid with a 2×2 square according to the rules:

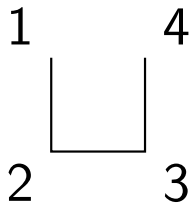


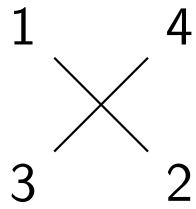




The LUX Method





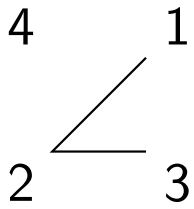


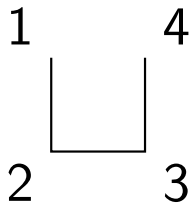
8	1	6
3	5	7
4	9	2

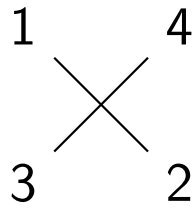
L	L	L
L	U	L
U	L	U

32	29	4	1	24	21
30	31	2	3	22	23
12	9	17	20	28	25
10	11	18	19	26	27
13	16	36	33	5	8
14	15	34	35	6	7

The LUX Method





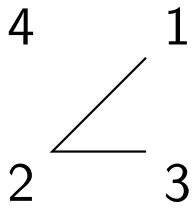


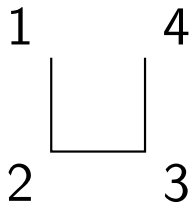
8	1	6
3	5	7
4	9	2

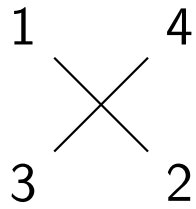
L	L	L
L	U	L
U	L	U

32	29	4	1	24	21
30	31	2	3	22	23
12	9	17	20	28	25
10	11	18	19	26	27
13	16	36	33	5	8
14	15	34	35	6	7

The LUX Method







8	1	6
3	5	7
4	9	2

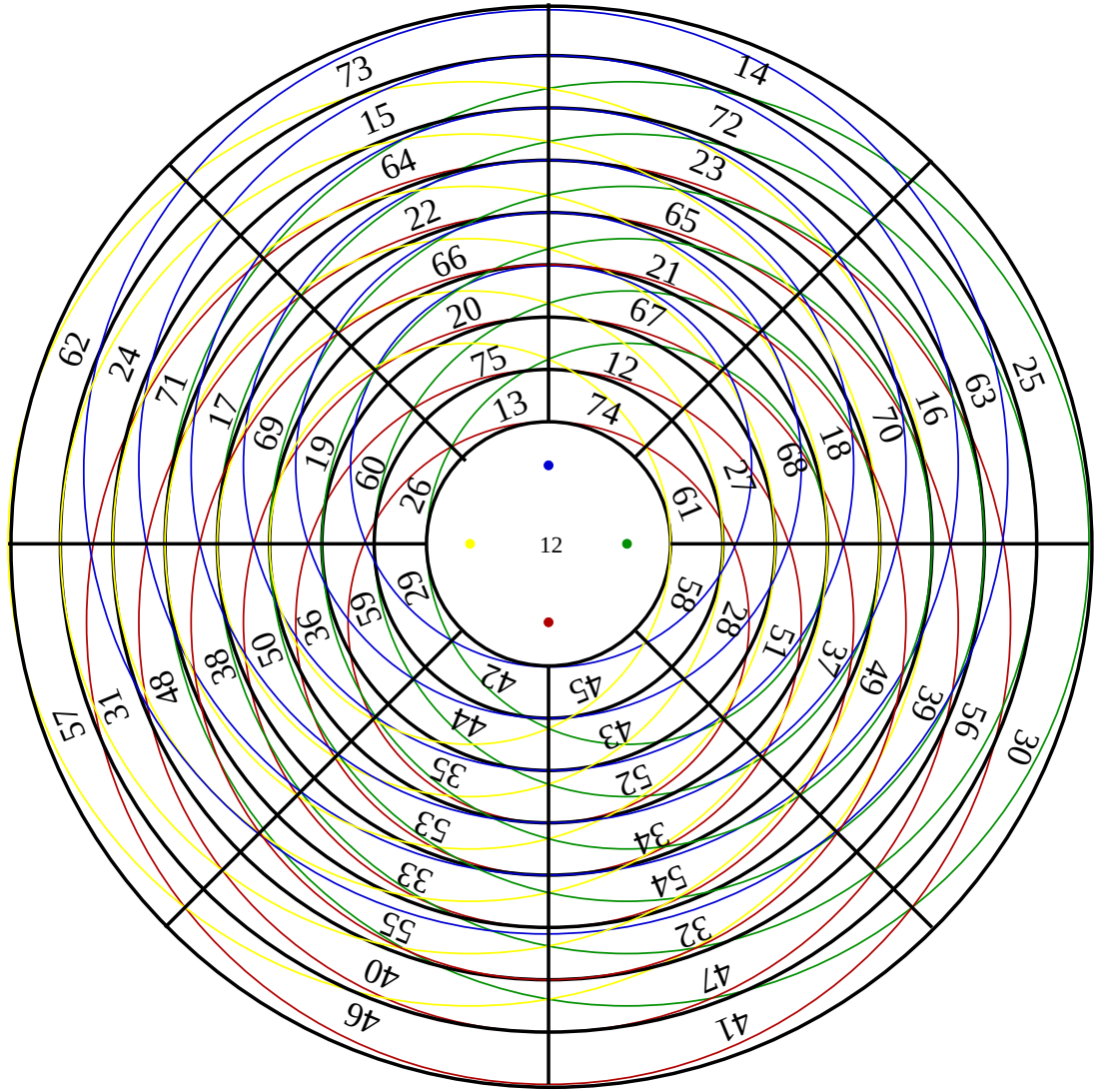
L	L	L
L	U	L
U	L	U

32	29	4	1	24	21
30	31	2	3	22	23
12	9	17	20	28	25
10	11	18	19	26	27
13	16	36	33	5	8
14	15	34	35	6	7

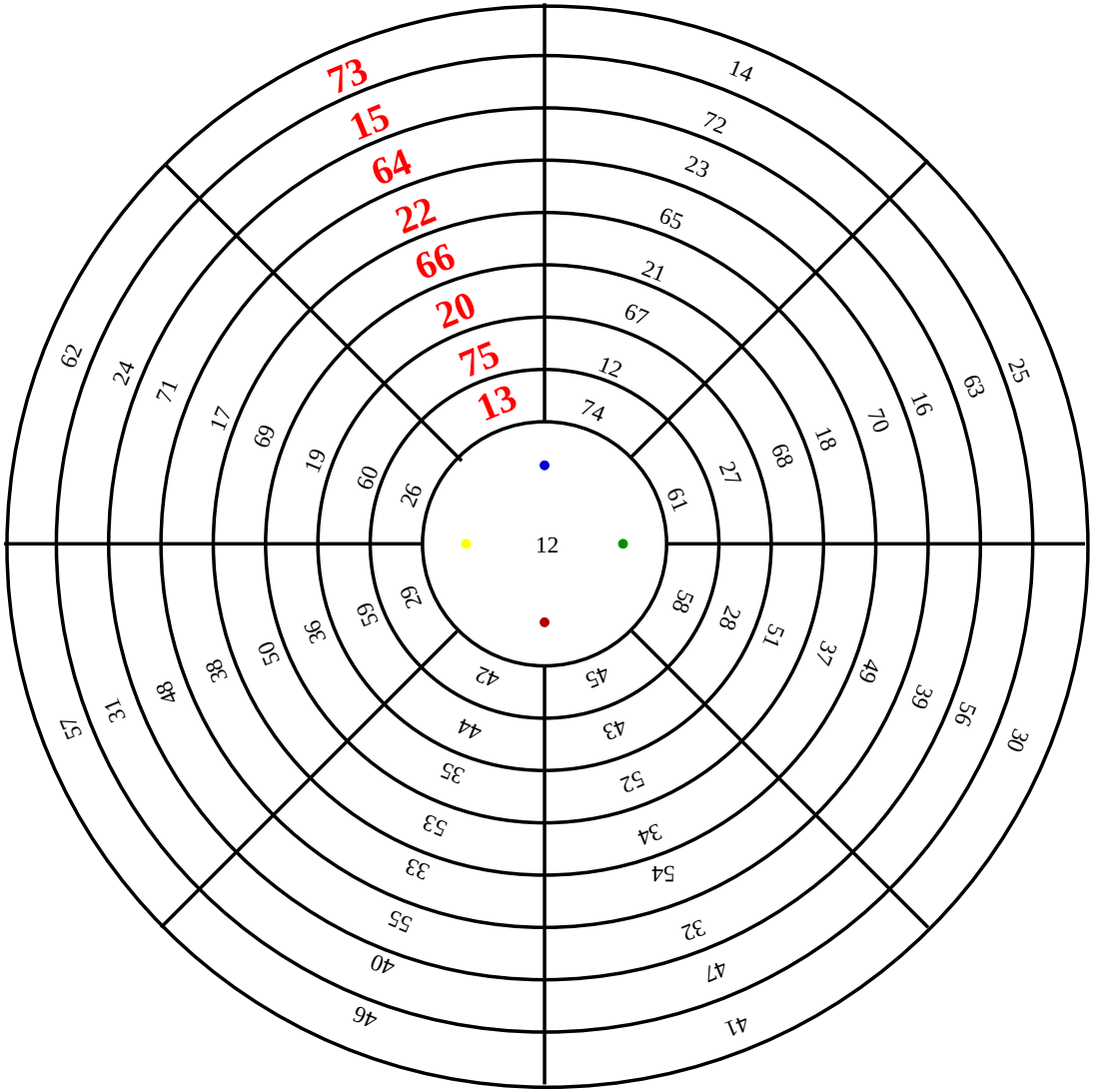
Ben Franklin's Magic Circles

"Dear Sir, As you seemed desirous of seeing the magic circle I mentioned to you, I have revised the one I made many years since, and with some improvements, sent it to you." In a letter to John Canton, May 29, 1765.

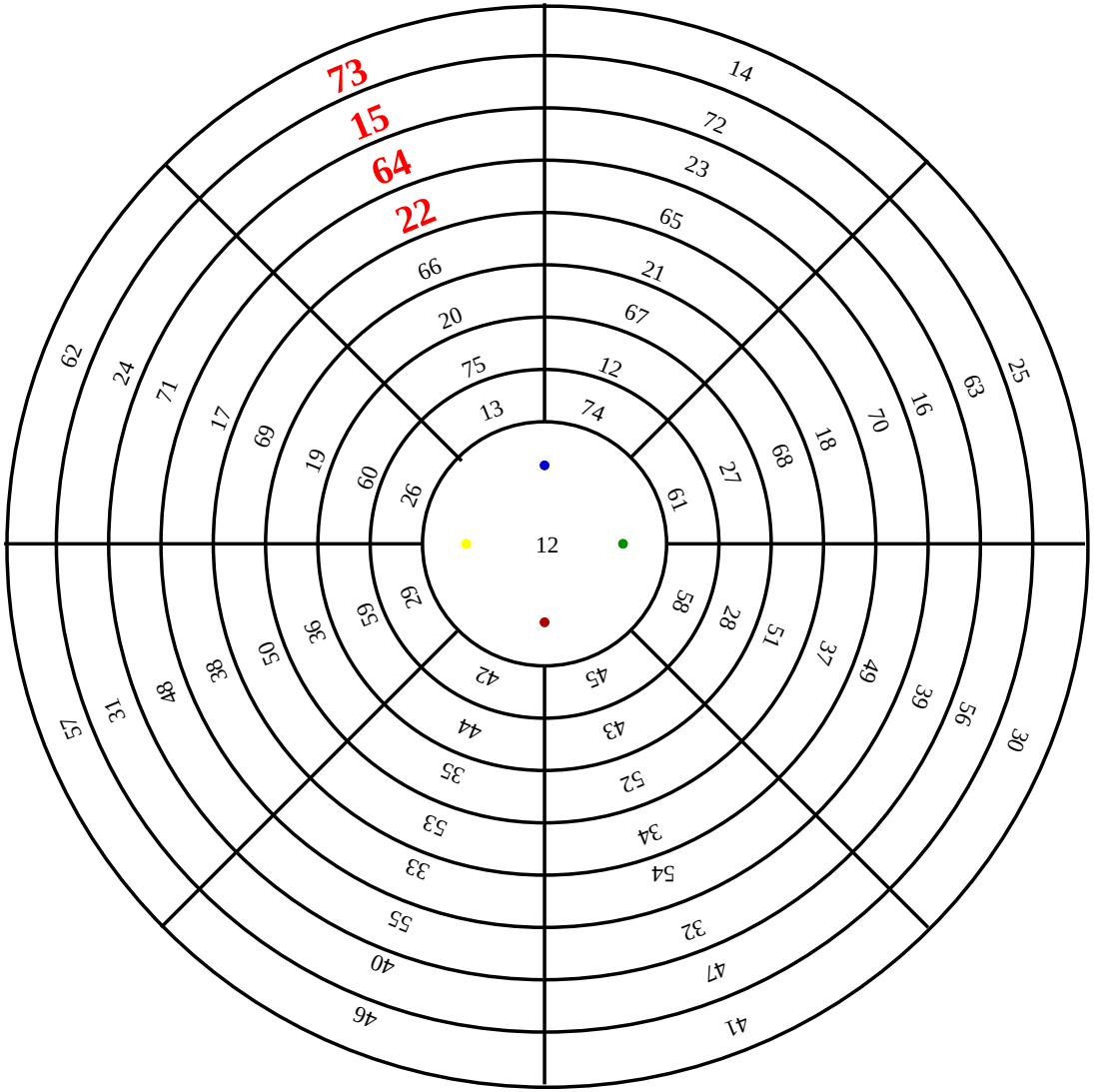
Benjamin Franklin's Magic Circle



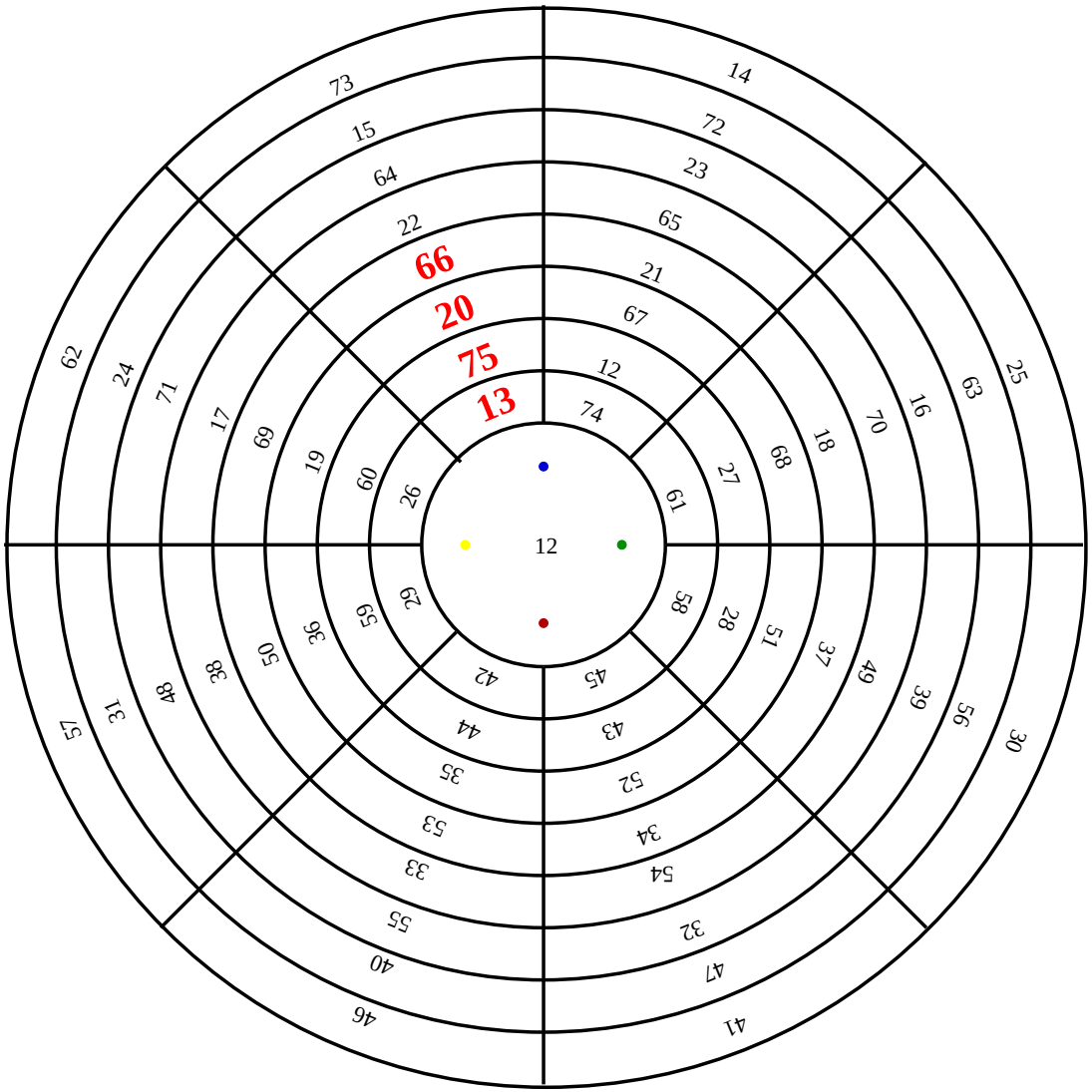
Radial Sum



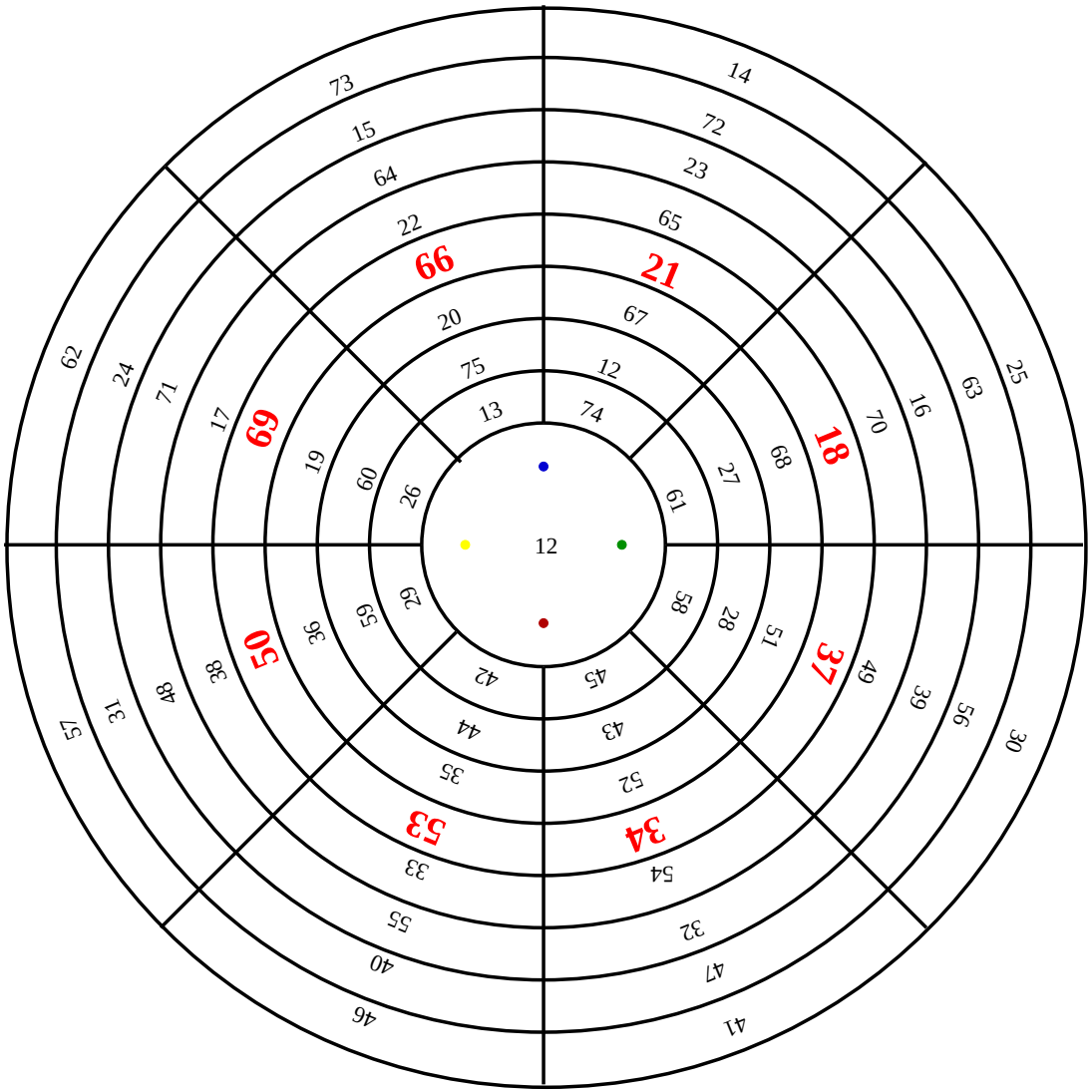
Outer-half Radial Sum



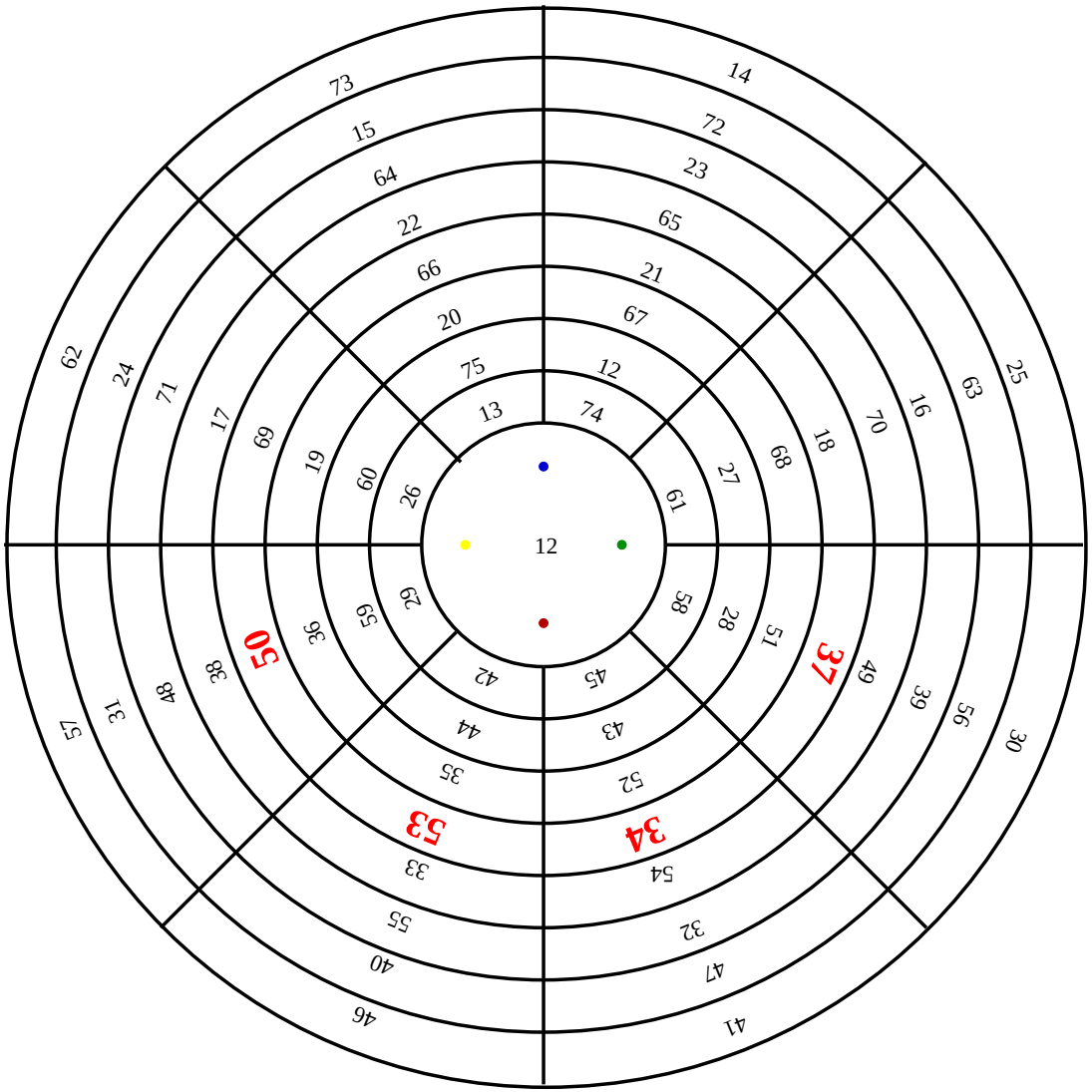
Inner-half Radial Sum



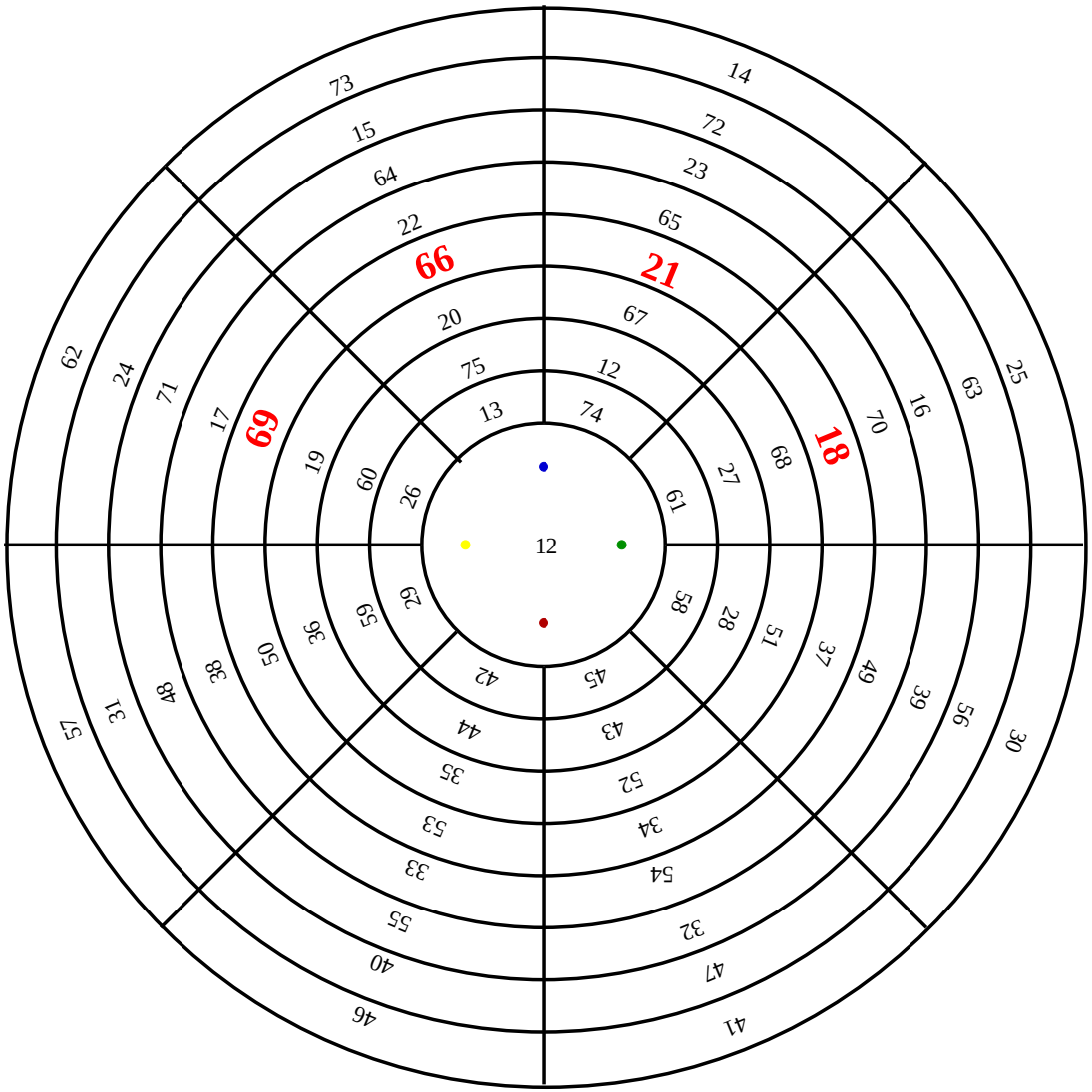
Annular Sum



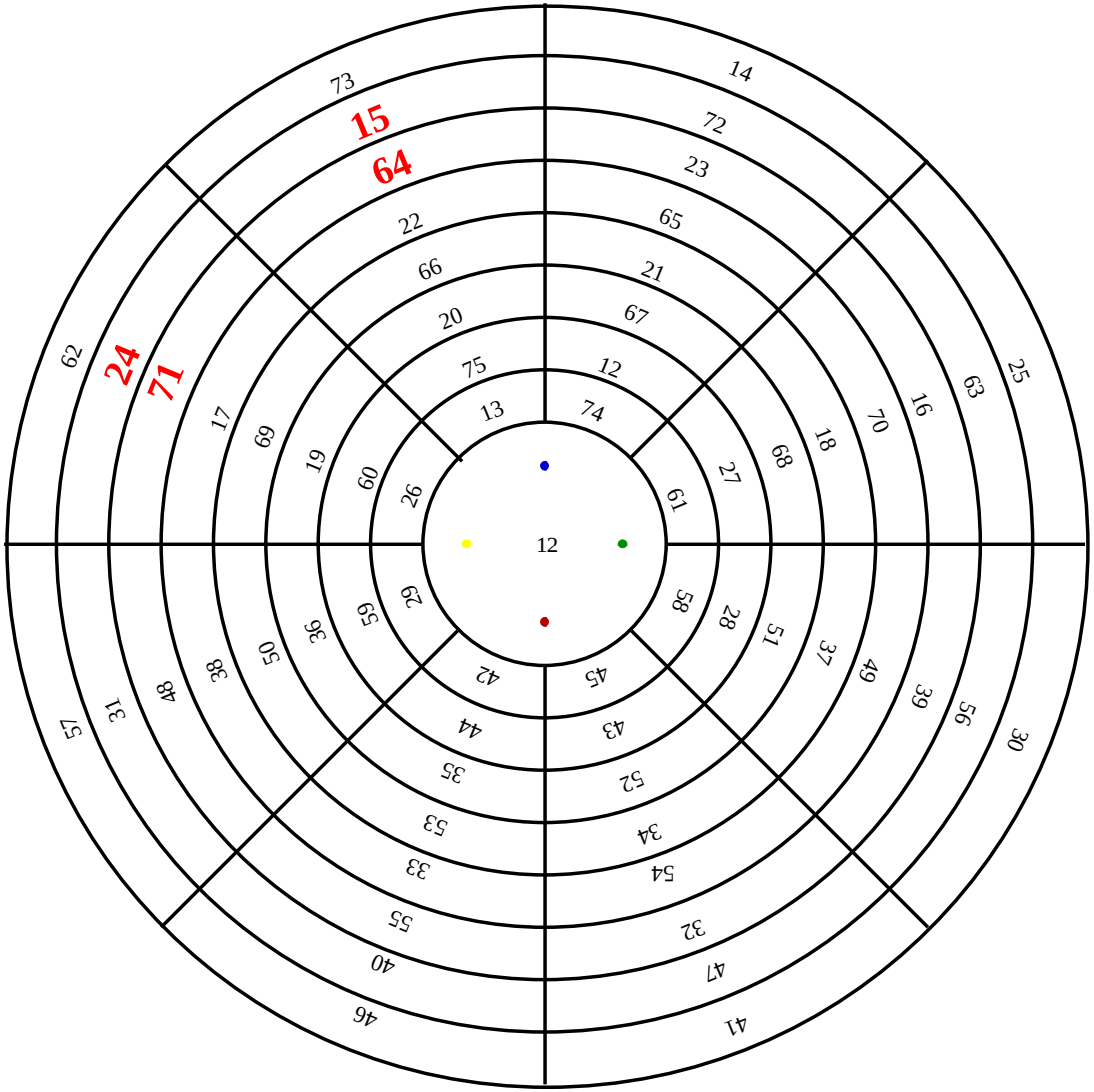
Lower-half Annular Sum



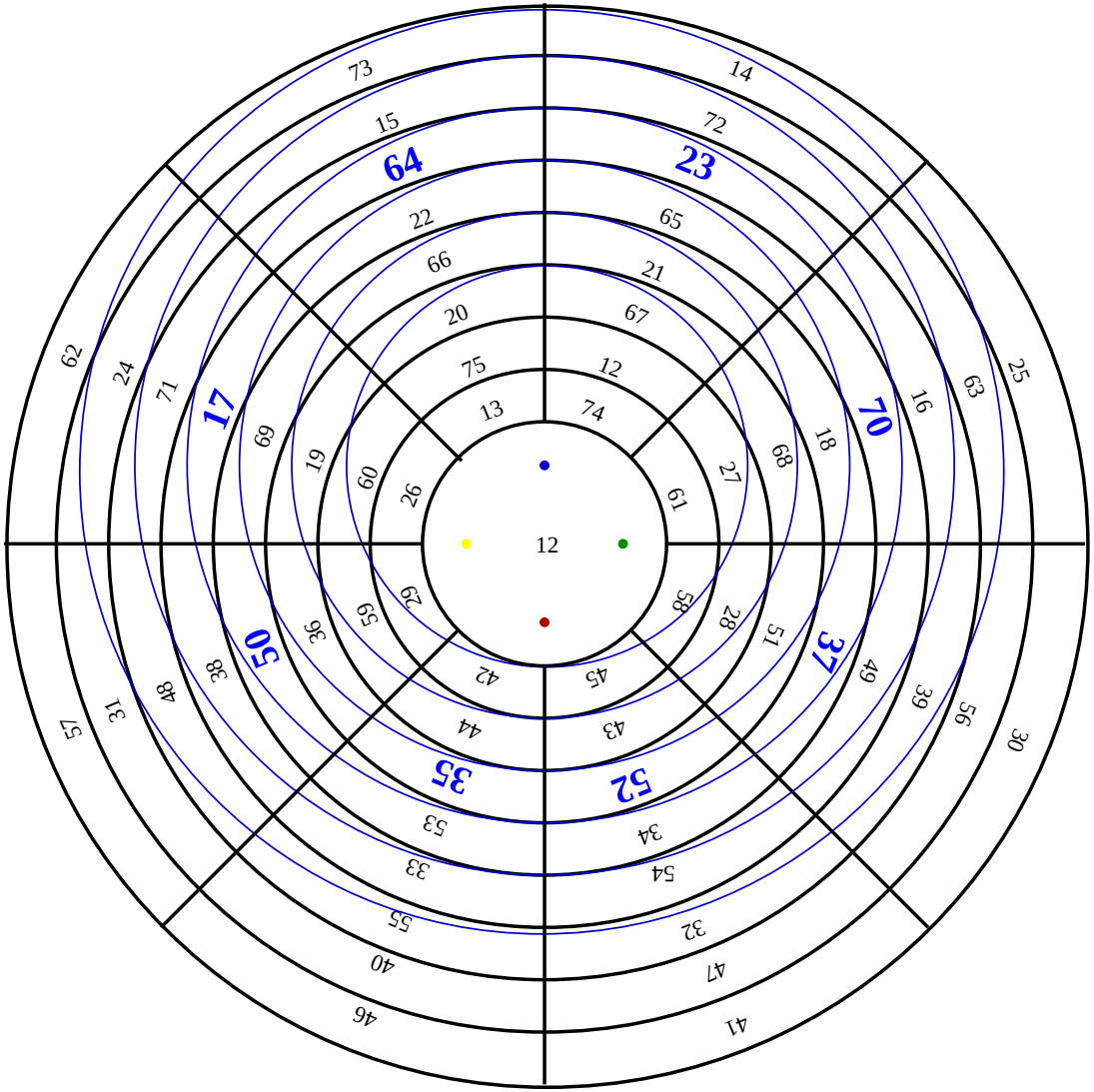
Upper-half Annular Sum



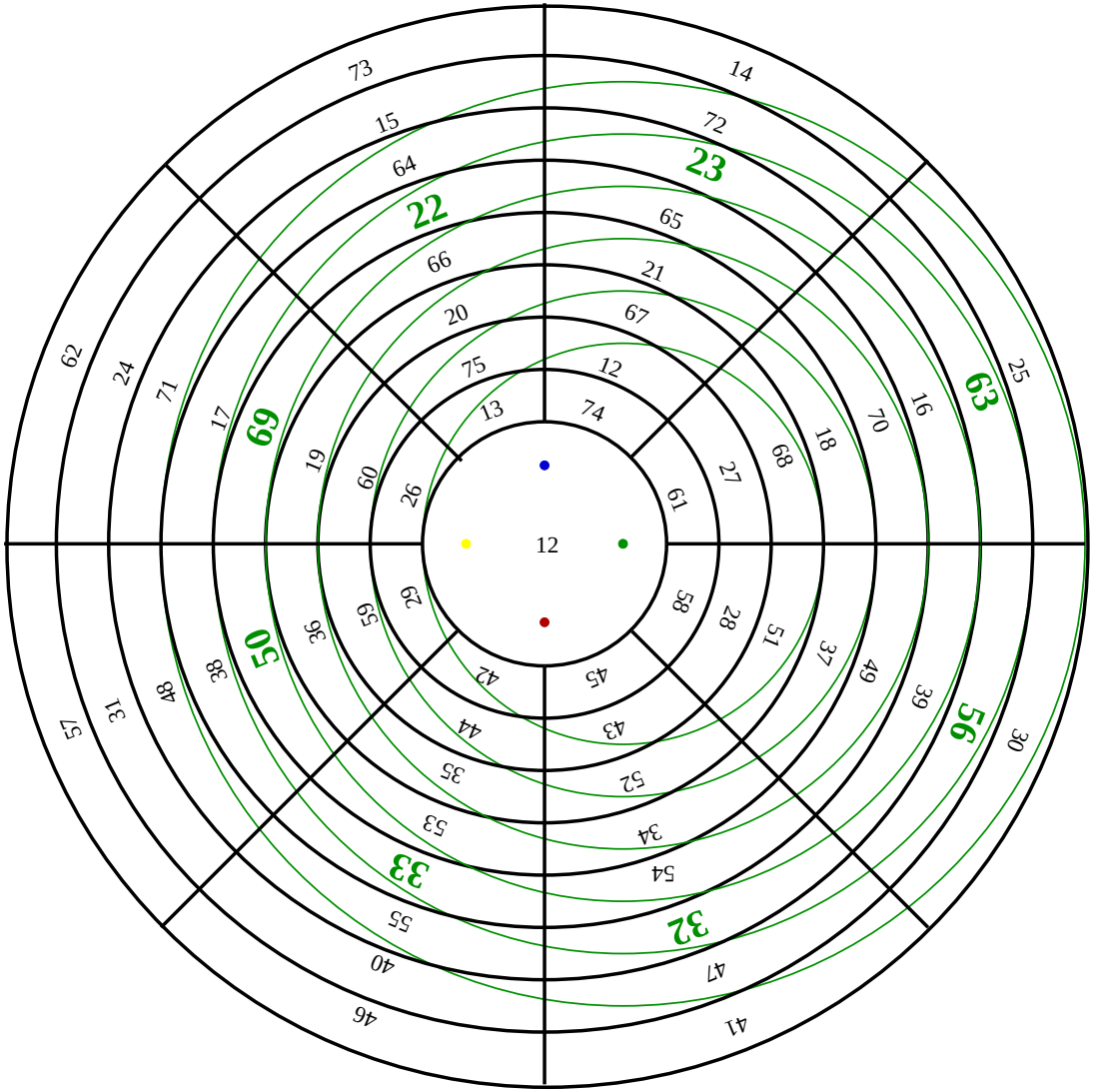
2×2 Block Sums



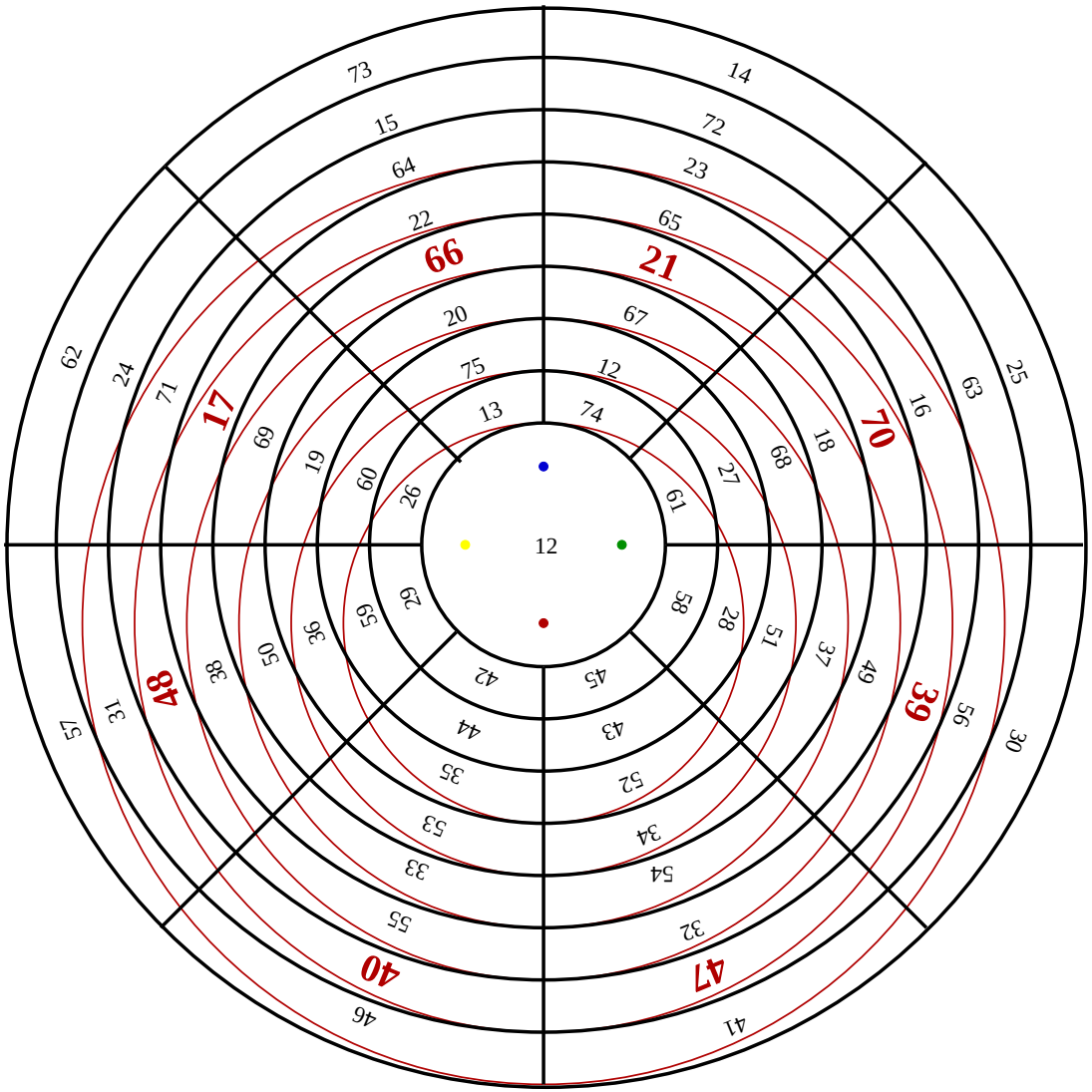
Northern Excentric Annular Sum



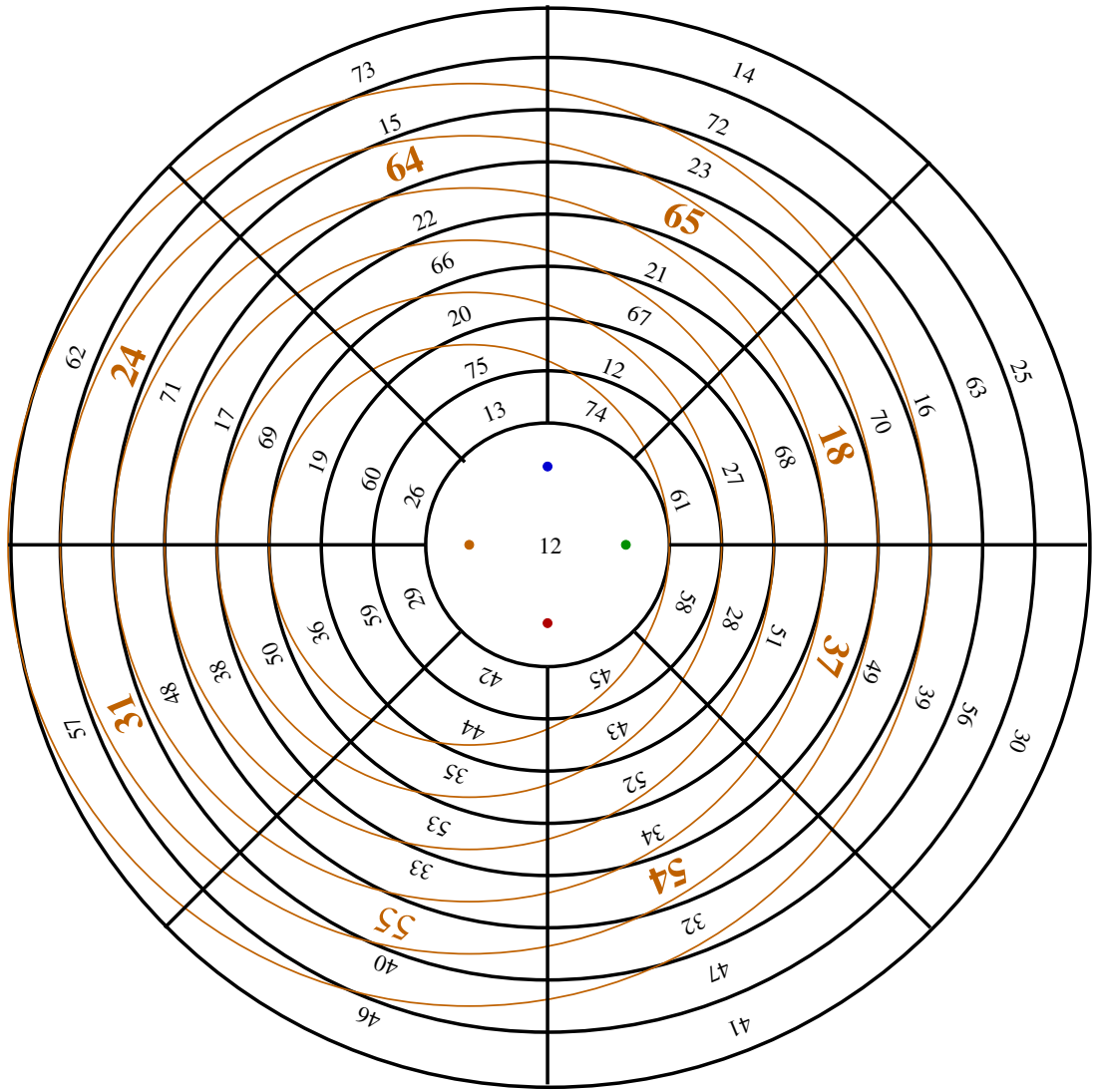
Eastern Excentric Annular Sum



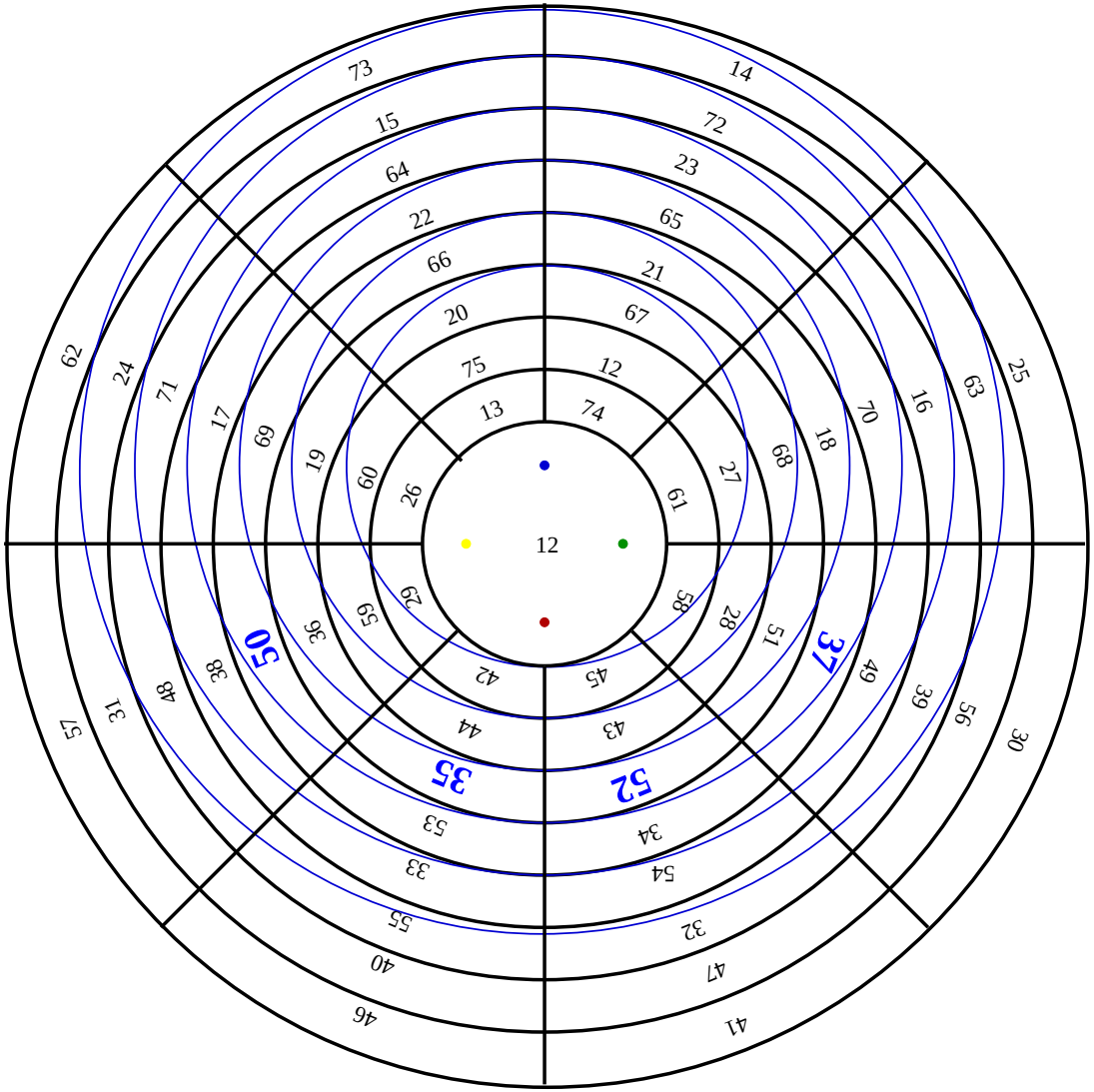
Southern Excentric Annular Sum



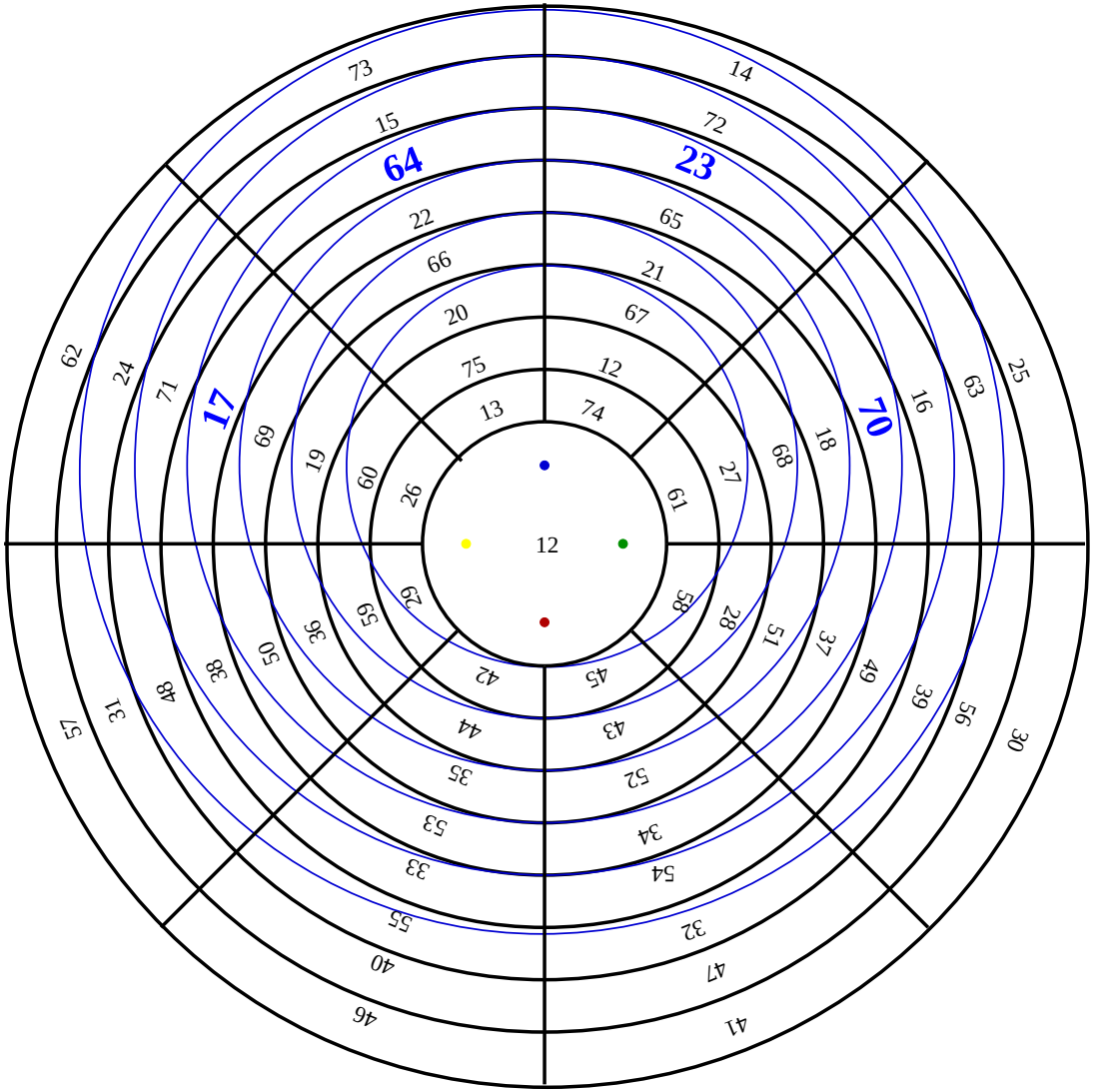
Western Excentric Annular Sum



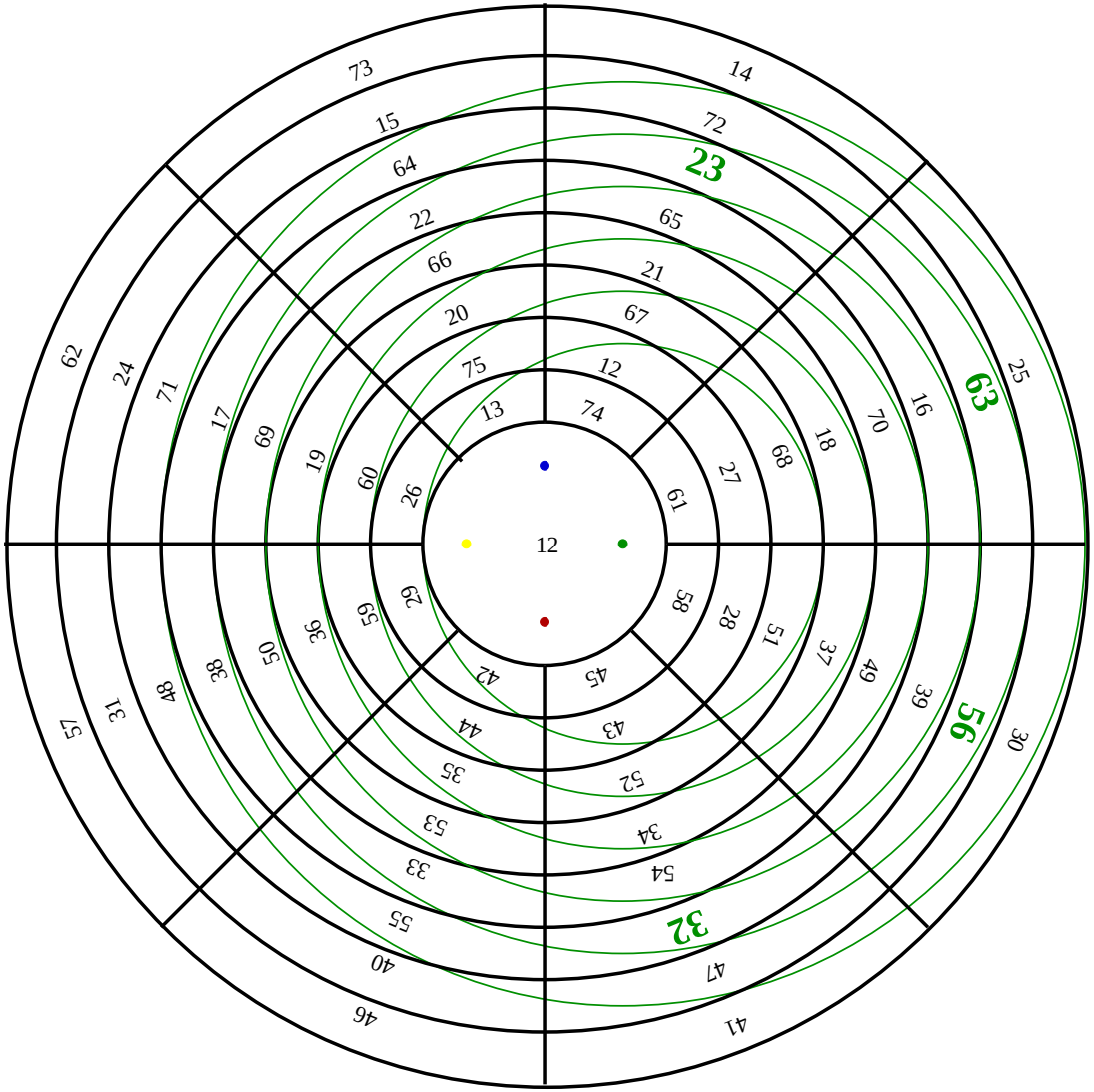
Vertically-centered Excentric Lower Half-annular Sum



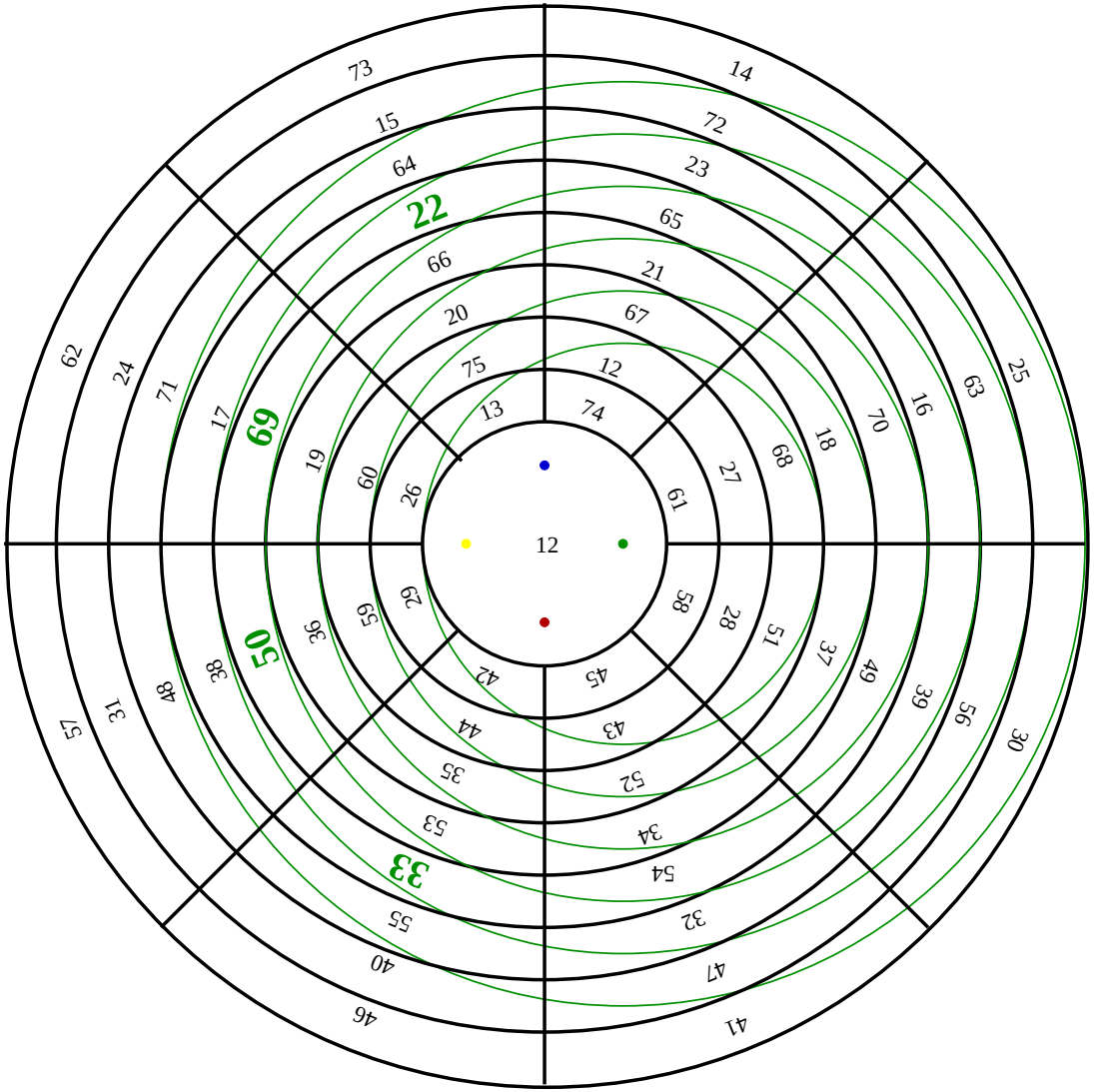
Vertically-centered Excentric Upper Half-annular Sum



Horizontally-centered Excentric Right Half-annular Sum



Horizontally-centered Excentric Left Half-annular Sum



Benjamin Franklin

"The magic square and circle, I am told, have occasioned a good deal of puzzling among the mathematicians here, but no one has desired me to show him my method of disposing the numbers. It seems they wish rather to investigate it themselves." In a letter to John Winthrop, July 2, 1768

Thank you!