Computer Proofs in Algebra, Combinatorics and Geometry

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Outline

1. Example of human proof.
2. Example of computer proof.
3. History of first major computer proof.
4. Recent results

Definition of “Proof”

Definition of “Proof”

- Proof: “An argument or evidence establishing the truth of a statement.”

- From Bing:
- Definitions of **proof** (n)
- proof [proof]
- **conclusive evidence:** evidence or an argument that serves to establish a fact or the truth of something
- **test of something:** a test or trial of something to establish whether it is true
- **state of having been proved:** the quality or condition of having been proved
- **Synonyms:** resistant, resilient, impervious, immune

Statement

There exists an infinite number of prime numbers.

True or False?

Human Proof

Statement: There exists an infinite number of prime numbers.

Proof: Assume there exists only N distinct primes:

$$1<p_1 = 2 < p_2=3 < p_3 < \ldots < p_N.$$

Set $K= p_1\ p_2\ p_3 \ldots p_N + 1$.

K has a prime factorization say $K= q_1\ q_2\ q_3 \ldots q_M$

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K has a prime factorization say $K= q_1 \, q_2 \, q_3 \ldots q_M$.

Subtracting

$$(*) \qquad q_1 \, q_2 \, q_3 \ldots q_M \; - \; p_1 \; p_2 \, p_3 \ldots p_N =1.$$

If $q_2 = p_j$ for some j then the left side of (*) is divisible by p_j with no remainder. But the right side of (*) is 1 so it is not divisible by $p_j >1$ with no remainder. Contradiction!

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If $q_1 = p_j$ for some j then the left side of (*) is divisible by p_j with no remainder. But the right side of (*) is 1 so it is not divisible by $p_j >1$ with no remainder. Contradiction!

Beautiful Proof !

Theorem: There exists an infinite number of prime numbers.

Proof: Assume there exists only N distinct primes:

$$1<p_1 = 2 < p_2=3 < p_3 < \ldots < p_N.$$

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Q.E.D.

Example of Computer Proof

Use symbolic algebra package like Maple or Mathematica.

Directions: Color the map below with as few colors as possible without choosing the same color for two adjacent states. Use letters like G for green or Y for yellow if you don't have colored pencils.

United States of America

A 4-coloring of the states

http://en.wikipedia.org/wiki/File:Map_of_USA_with_state_names.svg

Experiment

Try constructing a map for yourself which requires 5 colors.

Statement

- **"Every map of states/countries/counties etc can be colored using 4 colors such that no two adjacent states are given the same color. "**

- True or False?

- Caveats: No two states touch at isolated points. Each state is connected.

Statement

"**Every map of states/countries/counties etc can be colored using 4 colors such that no two adjacent states are given the same color.** "

History:

- 1852: Conjectured to be true by Francis Guthrie (cartagrapher or botonist).
- Francis Guthrie -> Fredrick Guthrie -> Augustus De Morgan -> Arthur Cayley

History

- 1852: Conjectured to be true by Francis Guthrie
- 1878: Cayley published Guthrie's conjecture.
- 1879: Kempe published a proof.
- 1880: Tait published a proof.
- 1890: Heawood pointed out a flaw with Kempe's proof!
- 1891: Petersen pointed out a flaw with Tait's proof!

….

- Many proofs and disproofs apear and get rejected. But much progress was made along the way.

History

- 1852: Conjectured to be true by Francis Guthrie
- 1878: Cayley published Guthrie's conjecture.
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- 1890: Heawood pointed out a flaw with Kempe's proof!
- 1891: Petersen pointed out a flaw with Tait's proof!
- ….   Many proofs and disproofs apear and get rejected. But much progress was made along the way.   The field of graph theory was born into mathematics.
- 1976 : Appel and Haken publish a highly controversial computer assisted proof.   NY Times refuses to mention it.

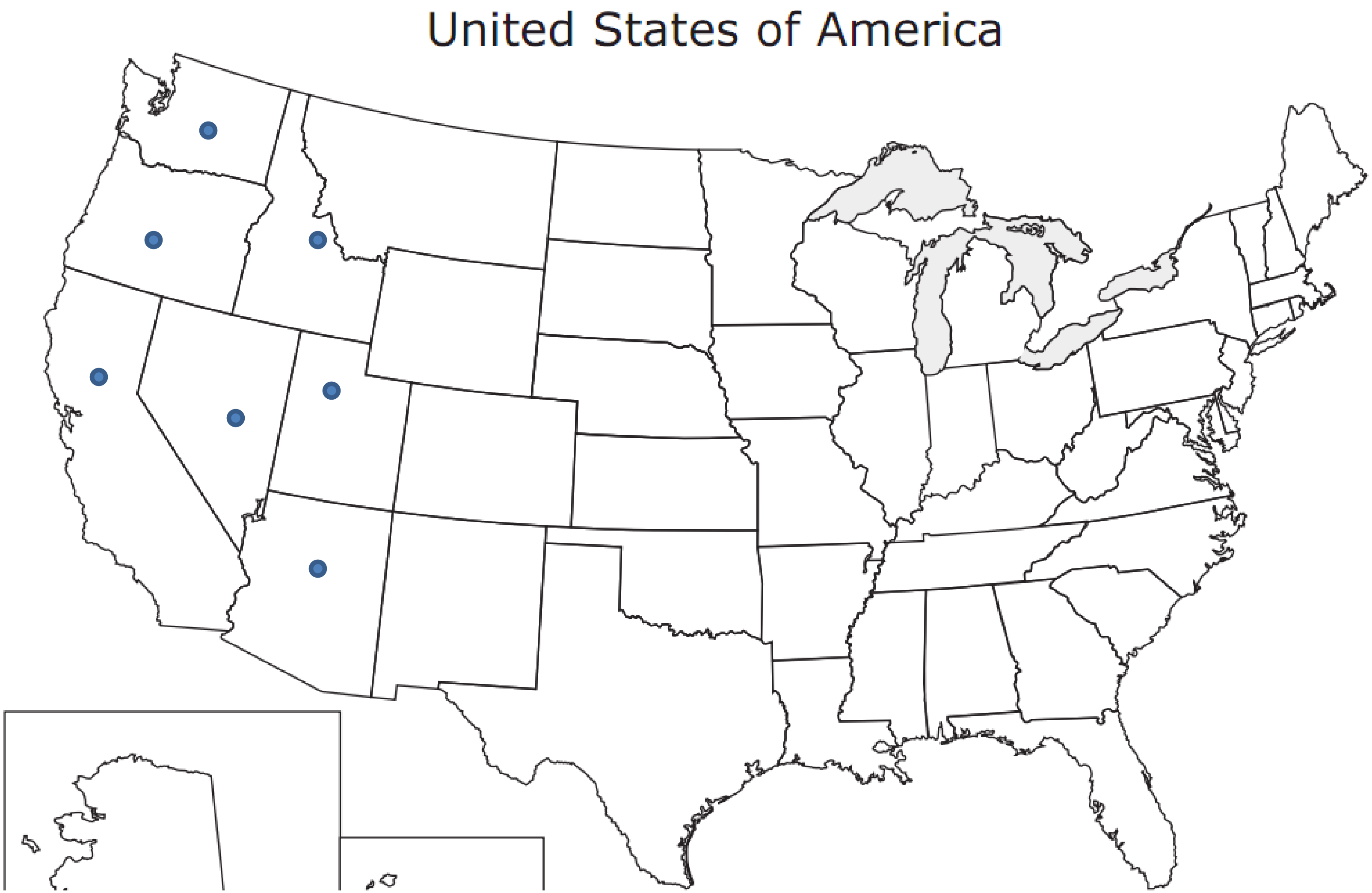
Reformulation

- Instead of coloring maps, the problem was generalized to coloring planar graphs.

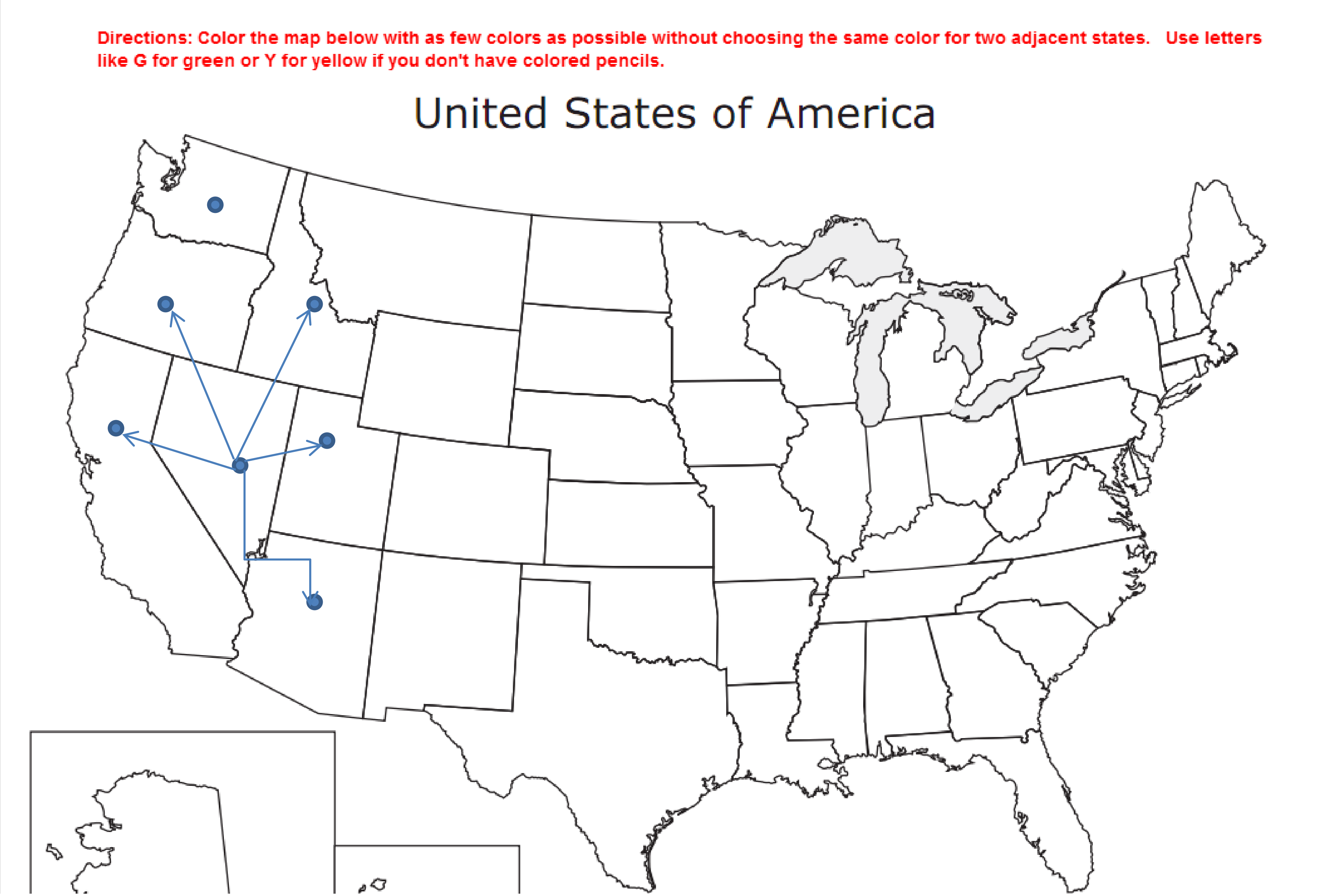
- Replace each state with a bold dot = vertex.
- Connect the dots representing two states if and only if they are adjacent on the map by path on the paper = edge.

Place a bold dot in each state.   Each dot is called a vertex of the graph.

Directions: Color the map below with as few colors as possible without choosing the same color for two adjacent states.   Use letters like G for green or Y for yellow if you don't have colored pencils.



Draw a path between every pair vertices representing adjacent states which Only passes through those two states. Each path is called an edge of the graph.



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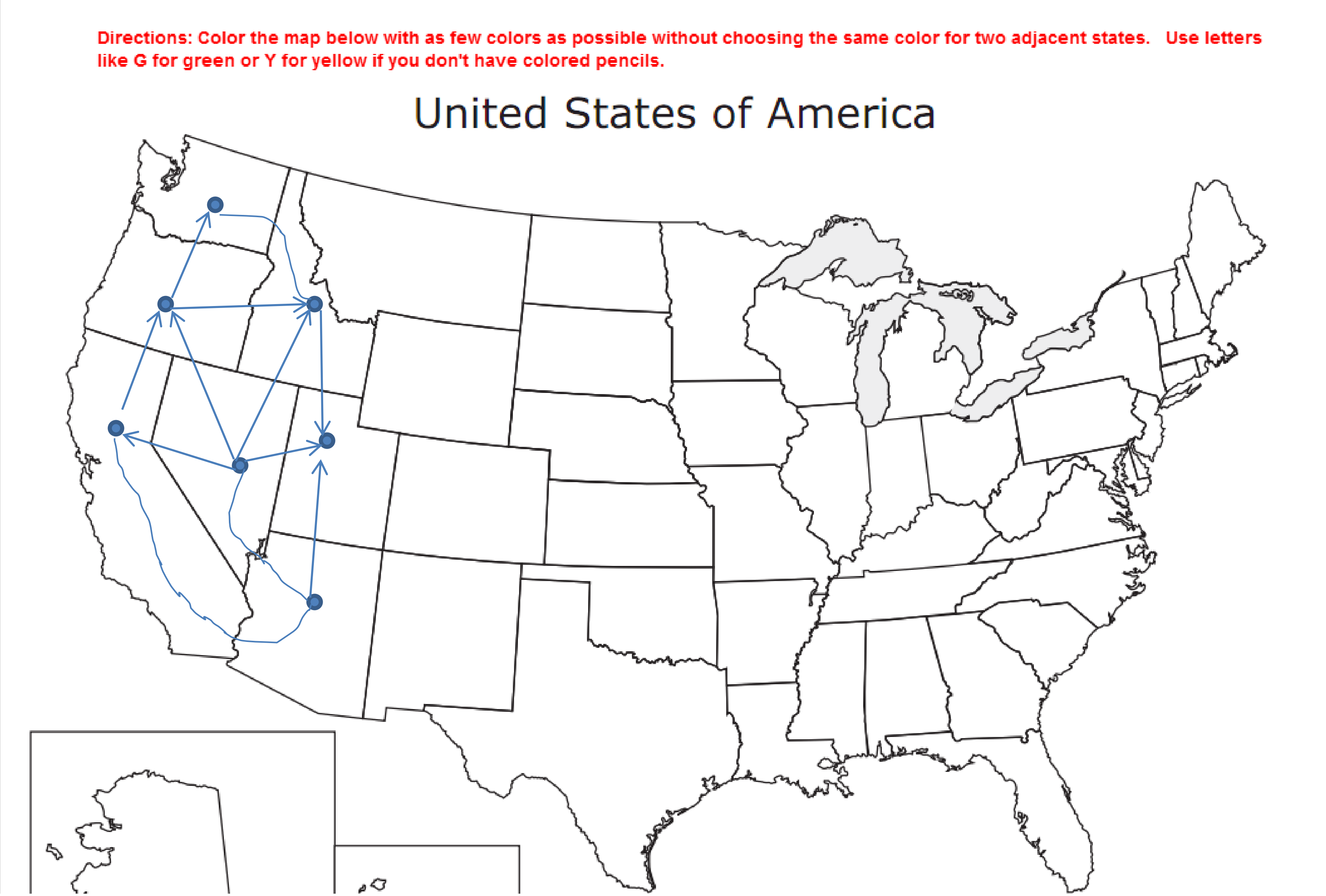
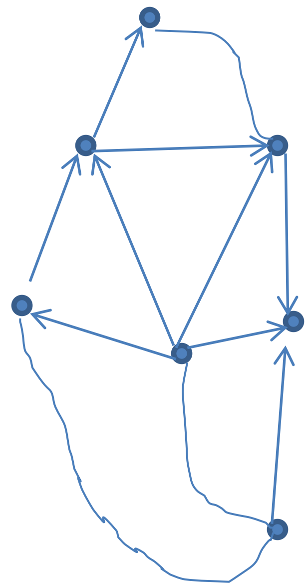


Delete the map. What remains is a planar graph.



A graph G=(V,E) is a set of vertices V and a subset of all pairs of vertices E.

G is planar if all the edges can be drawn in the plane without crossing each other.

Question: Can every map of states be represented by a planar graph?

Question: Is every graph planar?

Four Color Theorem

"Every vertex in a planar graph can be assigned a color distinct from all of its neighbors using at most 4 colors."

Controvercy over Computer Proof

- Imagine back to 1976… ..

PDP-8 Computer built around 1970

- Assembly Language (from Wikipedia: http://en.wikipedia.org/wiki/PDP-8)
- **
- This complete PDP-8 assembly language program outputs "Hello, world!" to the teleprinter.

```
  *10                 / Set current assembly origin to address 10,
  STPTR,    STRNG-1     / An auto-increment register (one of eight at 10-17)

  *200                 / Set current assembly origin to program text area
  HELLO,  CLA CLL       / Clear AC and Link again (needed when we loop back from tls)
        TAD I Z STPTR / Get next character, indirect via PRE-auto-increment address from the zero page
        SNA          / Skip if non-zero (not end of string)
        HLT          / Else halt on zero (end of string)
        TLS          / Output the character in the AC to the teleprinter
        TSF          / Skip if teleprinter ready for character
        JMP .-1       / Else jump back and try again
        JMP HELLO     / Jump back for the next character
  STRNG,  310          / H
        345          / e
        354          / l
        354          / l
        357          / o
        254          / ,
        240          / (space)
        367          / w
        357          / o
        362          / r
        354          / l
        344          / d
        241          / !
        0            / End of string
  $HELLO               /DEFAULT TERMINATOR
```

Controversy over Computer Proof

Appel and Haken Proof (1976).

- Human part of the proof is over 1000 pages long and no one else has ever been able to verify it. Many typos were found.
- Computer portion of the proof is written in assembly language and no one else has programmed it.
- 1478 graphs had to be coded by hand.

Question: Are you convinced they have a proof?

Good ideas in the Appel-Haken Proof

Outline of Proof:

Assume G is a counterexample to the 4CT with a minimal number of vertices.

- Reducibility (human only).
- Unavoidability (computer assisted).
- Algorithm for finding a coloring in G

Good ideas in the Appel-Haken Proof

Outline: Assume G is a counterexample to the 4CT with a minimal number of vertices.

- Reducibility: AH give a finite list of 1478 configurations in graphs. Each one of these cannot appear in G because if it did, they gave rules to replace G by a smaller graph that would still be planar and require more than 4 colors.

Good ideas in the Appel-Haken Proof

- Unavoidability: Every minimal counterexample to the 4CT must contain one of the configurations on the list.

A configuration is a small neighborhood in a graph. AH prove they only need to look at the second neighbors of each vertex and they bound the number of neighbors in each configuration. There are only a finite number of such graphs.

Good ideas in the Appel-Haken Proof

Outline: Assume G is a counterexample to the 4CT with a minimal number of vertices.

- Reducibility: AH give a finite list of 1478 configurations in graphs which cannot appear in G.
- Unavoidability: G must contain one of the configurations on the list.

Question: What can you conclude about G?

Good ideas in the Appel-Haken Proof

- Reducibility (human only).
- Unavoidability (computer assisted).

Together imply there always exists a 4 coloring of any planar graph. But how do we find one?

- Algorithm for finding a 4-coloring in G. Guaranteed to succeed if a 4-coloring exists.

History

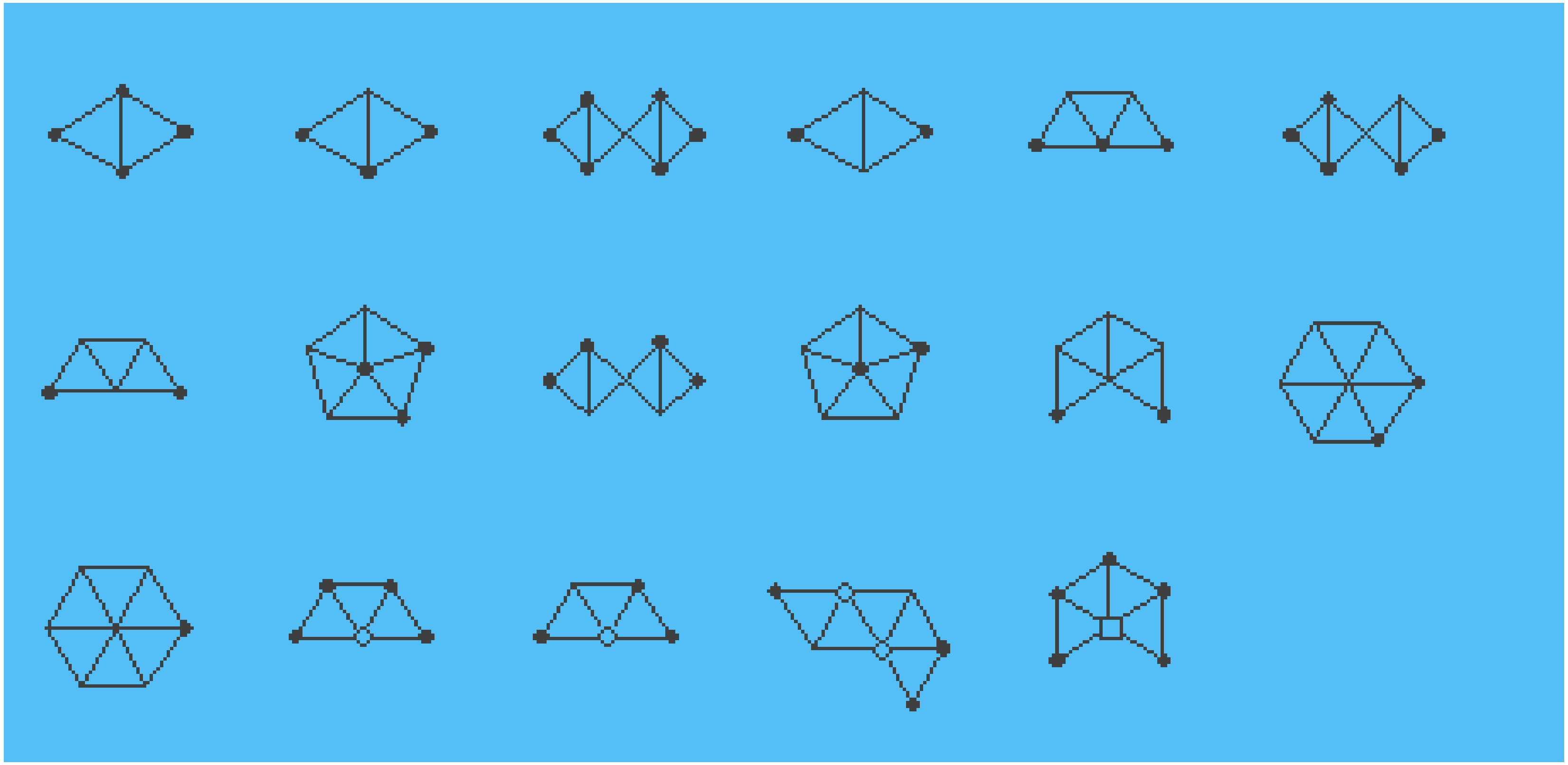
1996: “A New Proof of the Four Color Theorem”

Published by Robertson, Sanders, Seymour, and Thomas based on the same outline.

- Human part of the proof is about 20 pages.
- Computer portion of the proof was originally written in C and several other people have independently programmed it.
- No graphs had to be coded by hand.
- Only 633 configurations used.

Question: Are you convinced they have a proof?

Some of the 633 Configurations



History

1996: “A New Proof of the Four Color Theorem”

Published by Robertson, Sanders, Seymour, and Thomas based on the same outline.

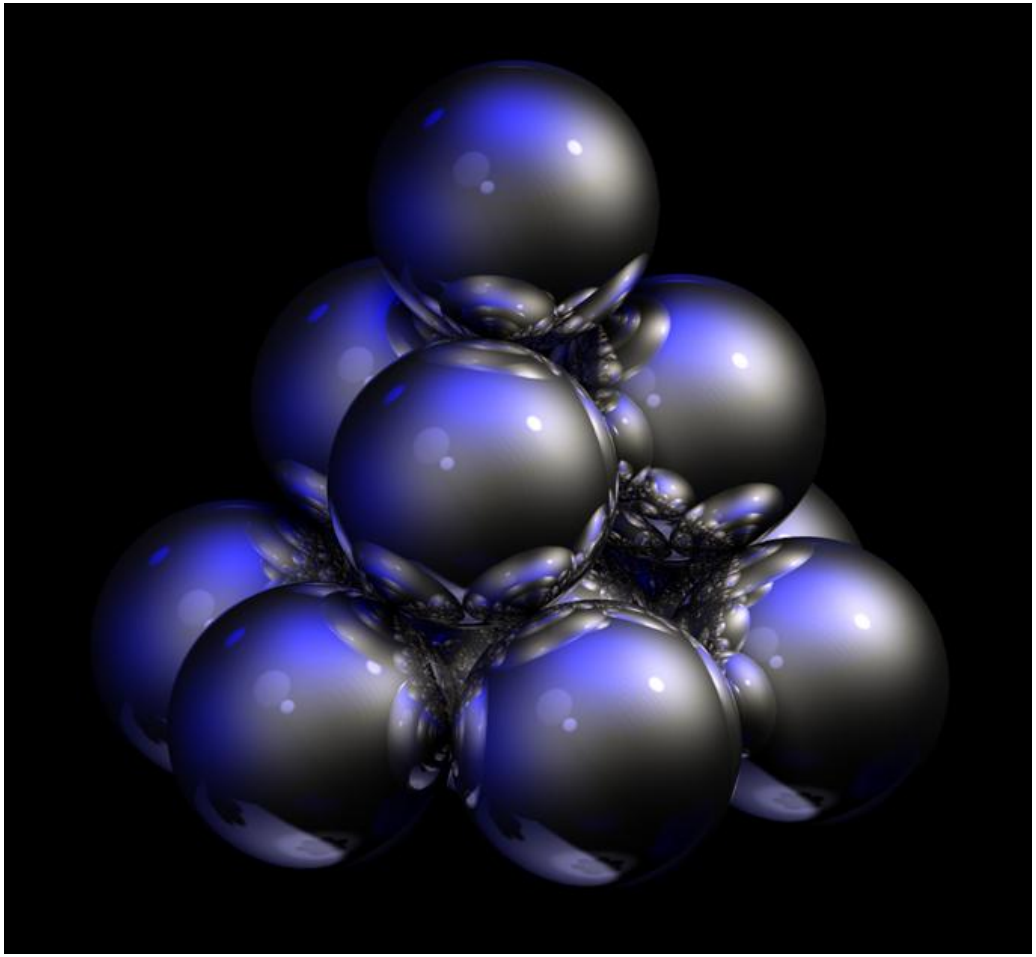
- Algorithm: RSST also give an algorithm to find a 4-coloring of a planar graph that takes about n^2 seconds on a graph with n vertices.

Kepler's Conjecture

Astronomers were wondering:

What is the best way to pack cannon balls in space so they are as close as possible?

Hexagonal Close Packing



http://upload.wikimedia.org/wikipedia/commons/8/8e/Close-packed_spheres.jpg

Kepler's Conjecture

What is the best way to pack cannon balls in space so they are as close as possible?

Conjecture: The portion of space filled by cannonballs in the densest possible packing is given by the hexagonal close packing and has density

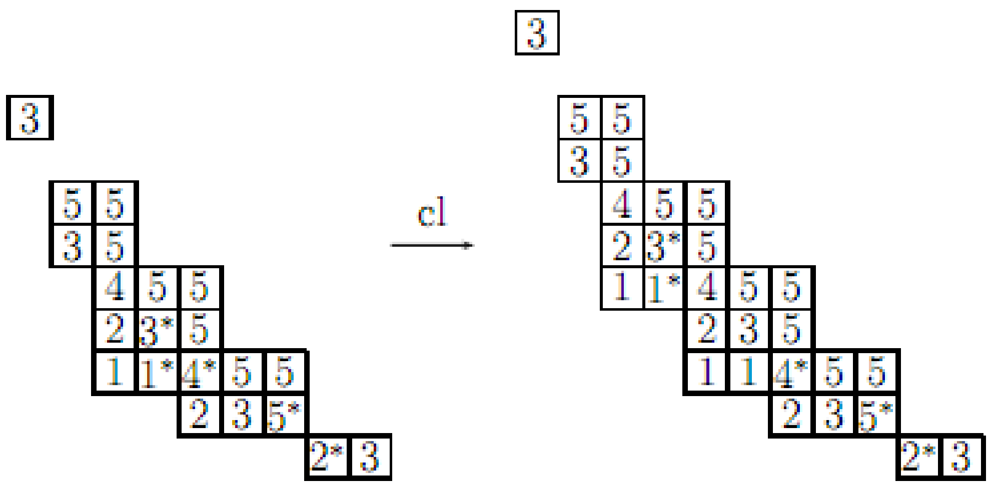
$$\frac{\pi}{\sqrt{18}} \approx 0.7404804898.$$

History of Hales Proof

- 1953: Toth showed that the problem could be reduced to a finite check of about 5000 cases.
- 1992: Thomas Hales and Samuel Ferguson began using linear programming to check the density of these cases.
- 1996 : Hales announced the proof was complete.
- 2005: Hales' paper was published after being reviewed by a committee of 12 referees who said they were 99% certain it was correct.

A halting problem

Problem: Find all graph types corresponding with rank 5 starred strong tableaux under cloning.



Human part of the proof is 50 pages long. It is ready to publish.
Computer part has been running since January, but hasn't finished.

Questions: When should we submit it for publication?
Do you think it will be controversial?

Philosophical Question

What is the value of a computer proof?

- We get a new result which we can build on!
- We learn one more method of using computers to prove theorem.
- Every computer proof with no human proof contains a miracle which makes it computable!

Summary

Computers can be very helpful proving theorems about…

- Algebraic identities.
- Finite calculations.
- Halting problems

And what else?

Lots more

- Origami: Can you fold this? See "Geometric Folding Algorithms" by Demaine and O'Rourke.

- Automatic Theorem Checking: Is this human proof correct? See "How to Write a Proof" by Leslie Lamport in *American Mathematical Monthly* 102, 7 (August-September 1993) 600-608.

- Game Theory: Does this game have a winning strategy? See history of Connect Four in Wikipedia.