MONTHLY MATH HOUR AT UW

COUNTING STARS

March 2025

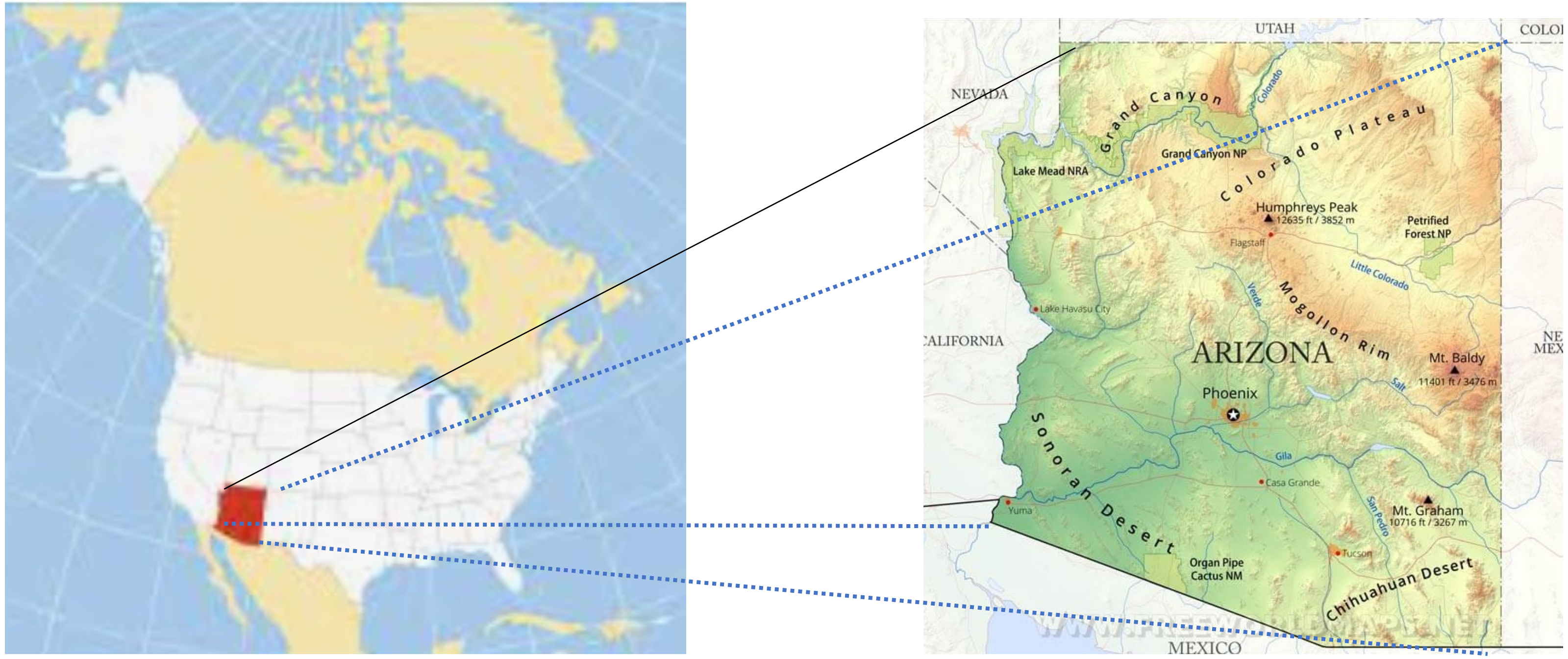
Mark Saul, Ph.D.

HOW MANY OF YOU KNOW WHAT THE WORD ‘LOGARITHM’ MEANS?

(Just asking. You don’t *have* to know for most of this talk!

COME WITH ME TO ARIZONA…

THE
‘GRAND
CANYON’
STATE

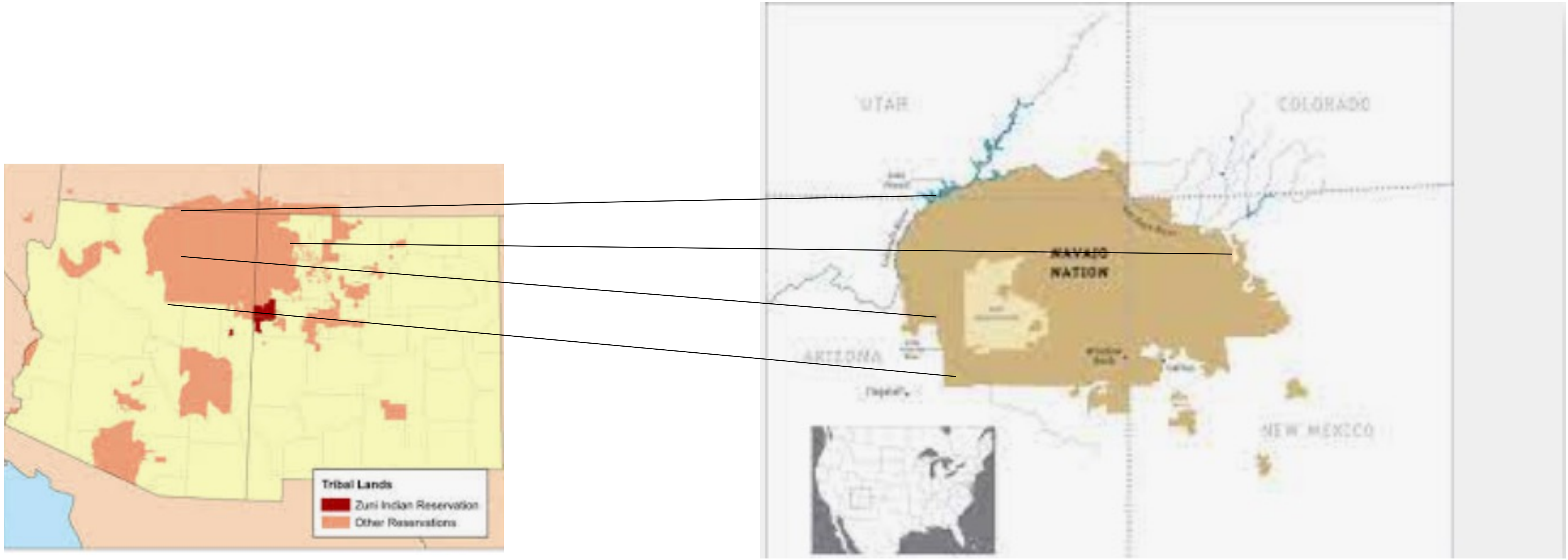


COME WITH ME TO ARIZONA…

THE
‘GRAND
CANYON’
STATE

BUT YOU CAN VISIT THE GRAND CANYON ANOTHER TIME.

TODAY WE WILL GO EAST FROM THE GRAND CANYON, TO THE LAND OF SOME NATIVE AMERICANS: THE HOPI, ZUNI, AND NAVAJO PEOPLE.



THE HOPI

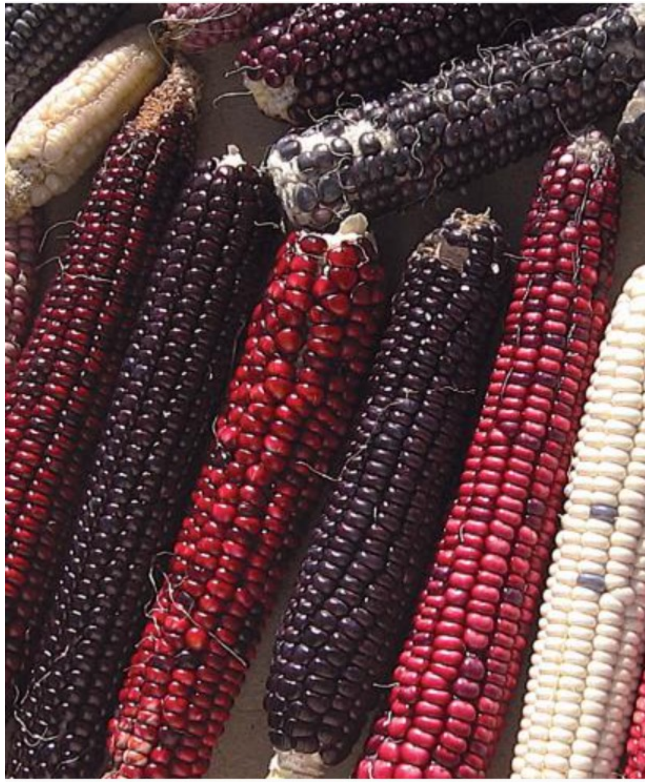
About 7000 Hopi people live on the Hopi reservation. It is desert country, but they have farmed the land for more than 1000 years.

Some of their villages are the oldest continuously settled towns in North America. Old Oraibi has been inhabited since at least 1000 AD.

Today, they are educating their young people for the modern economy.

THE HOPI

Their agricultural way of life has led to the development of a complex religion, incorporating celebrations of the seasons for planting and harvesting.



Hopi Elementary School

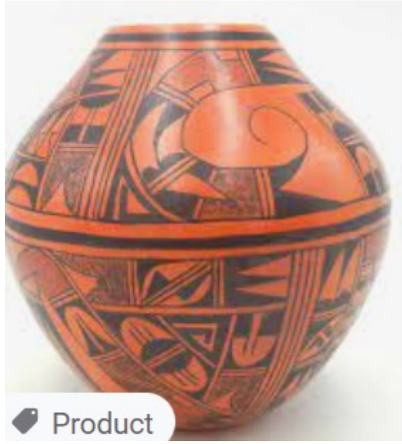
THE HOPI

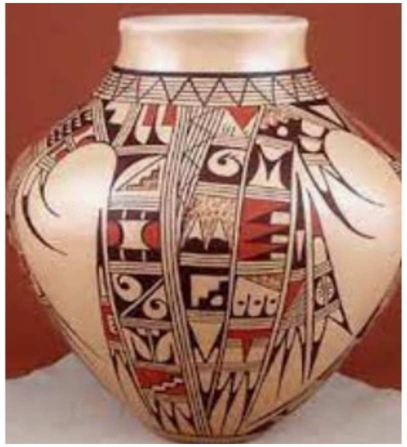
Their religion includes the existence of *kachinas*, spirits who help the Hopi survive in their harsh environment. Hopi children learn about these spirits through small statues, which are sometimes called “kachina dolls”.

But to the Hopi these are much more than playthings.

THE HOPI

The Hopi are also famous for their ceramics. They have made pots and jars out of clay for hundreds of years. More recently, they have developed ceramics as works of art.



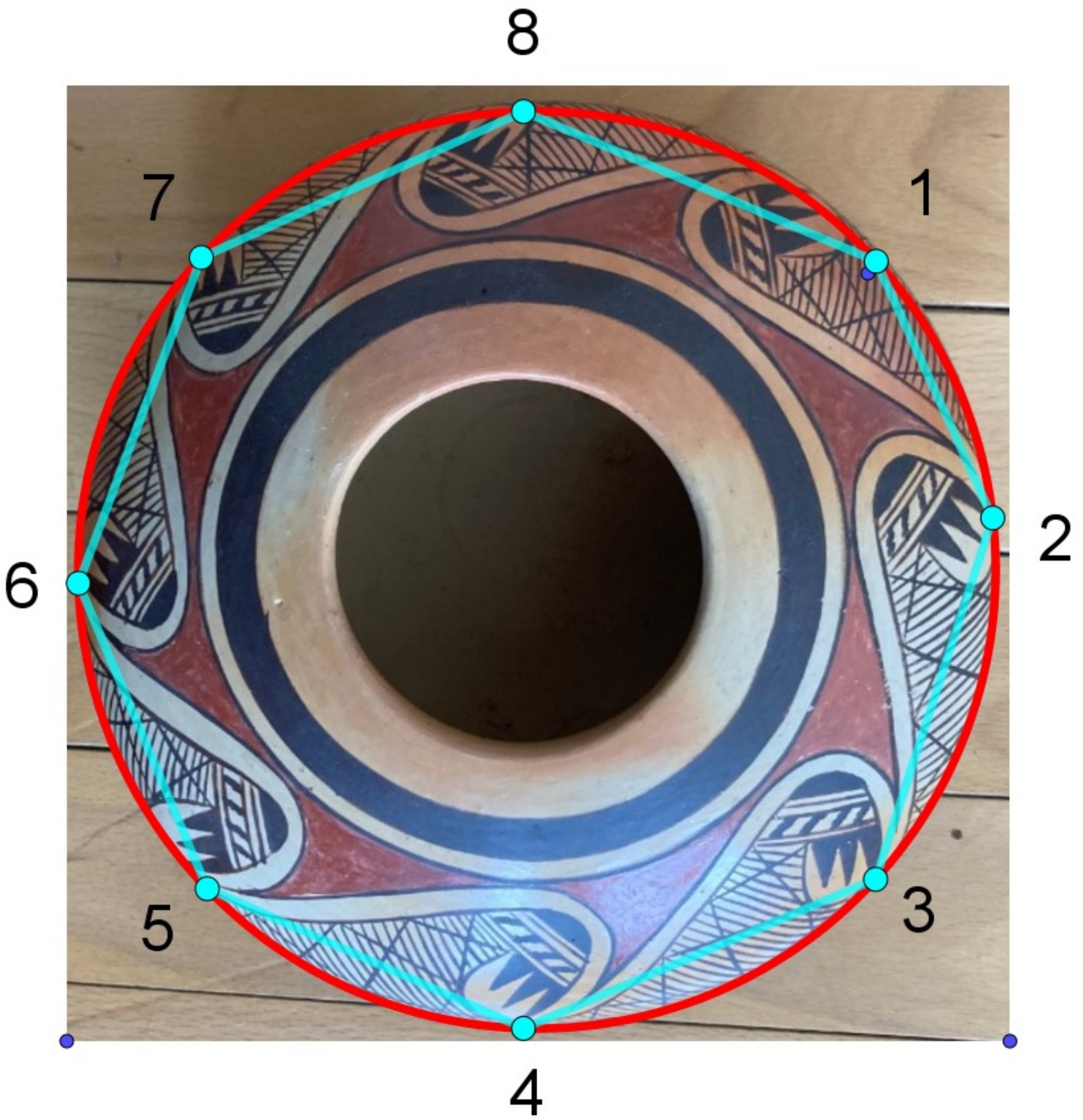



This is a pot by Fannie Nampeyo, a famous Hopi potter.

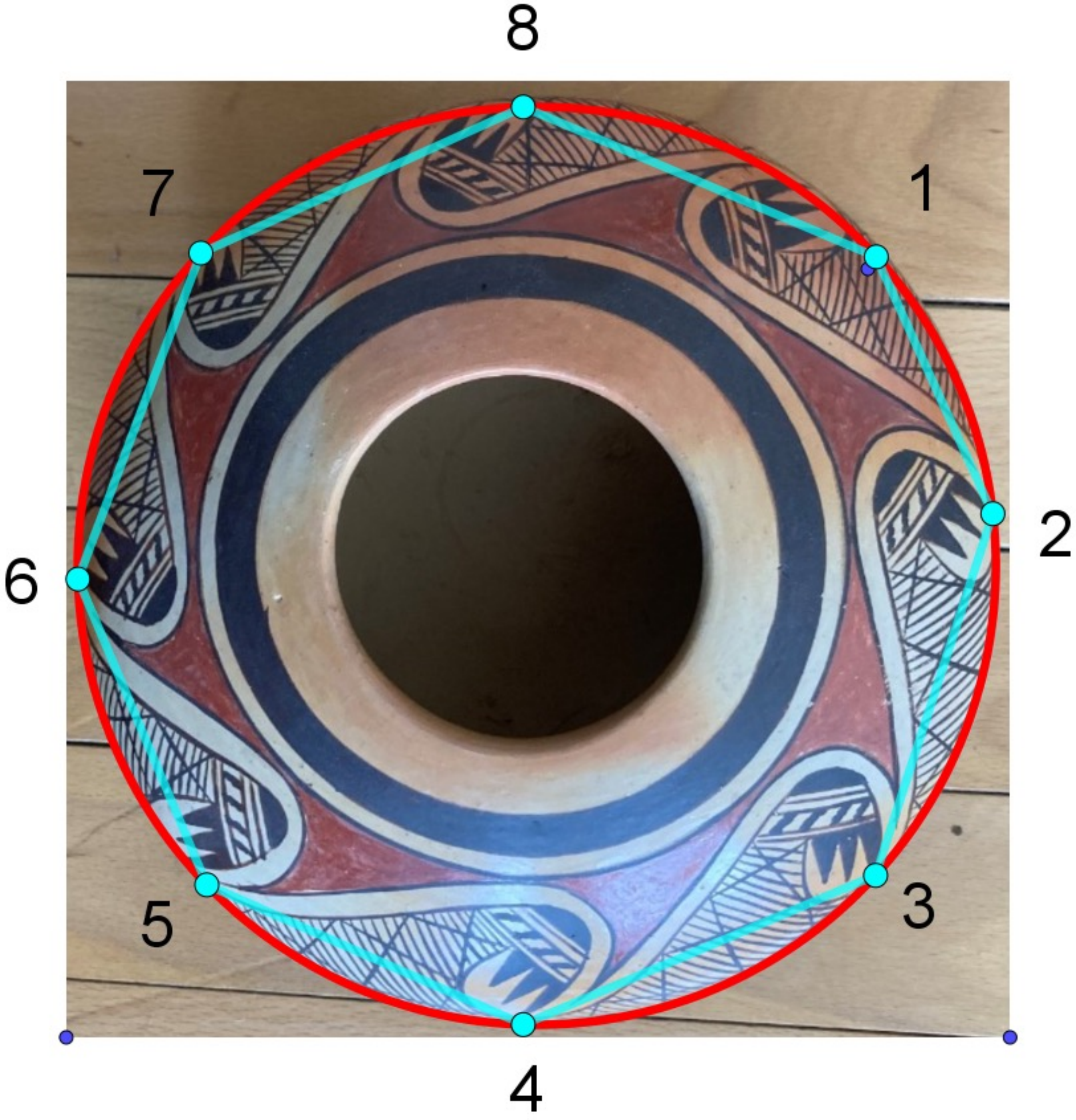
This is the same pot viewed from above. It has eight copies of the 'bear claw' motif laid out in a circle.

If you know a potter, you can ask her or him how to get the pot to be a circle, and how to divide it into eight equal parts for the repeated design. Those are all interesting mathematical questions. But we will ask some other questions.

Here we have drawn a (red) circle around the outside of the pot, and put a (blue) point at the same claw of each bear claw motif. That makes eight points equally spaced around the circle.

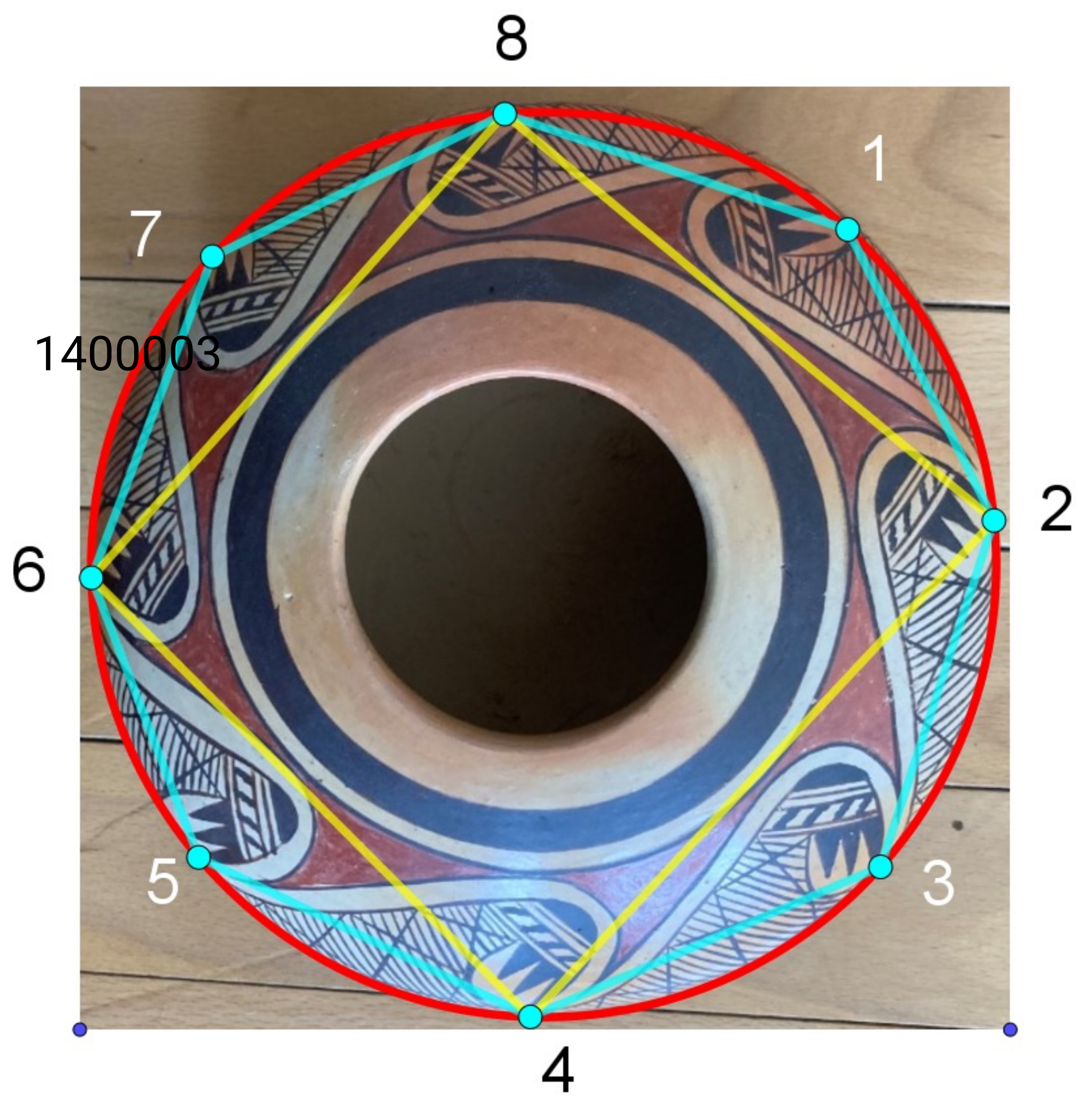


Now we’ve numbered the points and connected them to form an octagon.

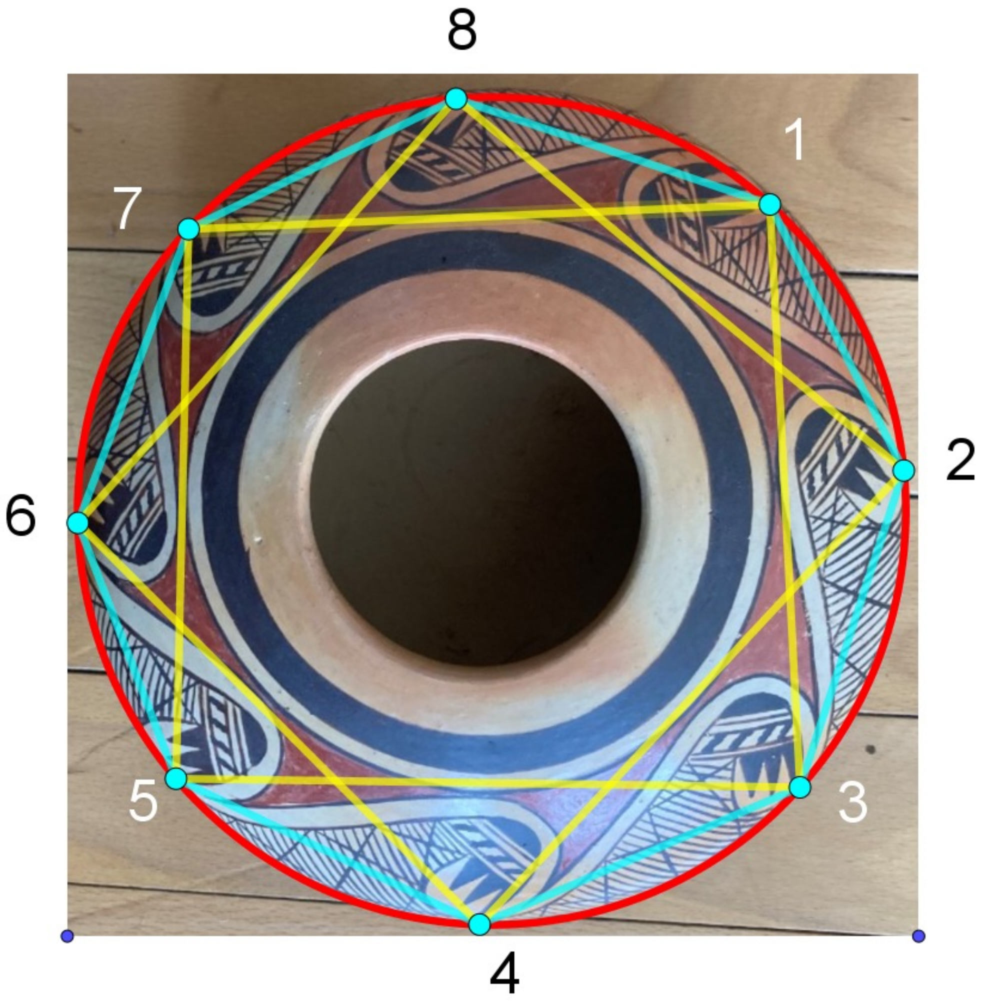


That is, we have started at the point number 8 (the highest number), and gone around to 1, then to 2, then to 3…., then back to 8.

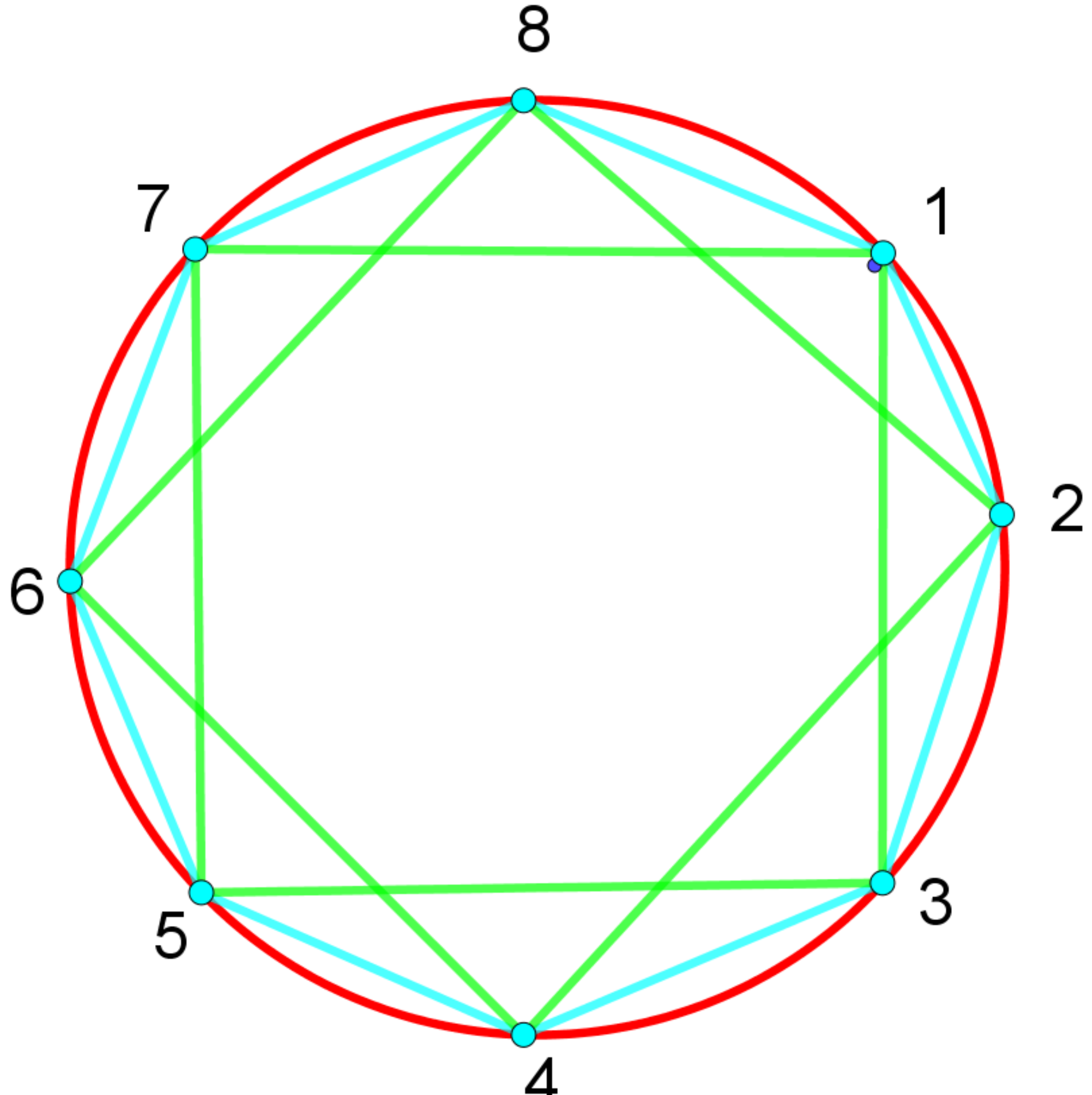
But what if we started at 8 and went to 2, skipping a number. Then kept skipping a single number? We would get a (yellow) square:



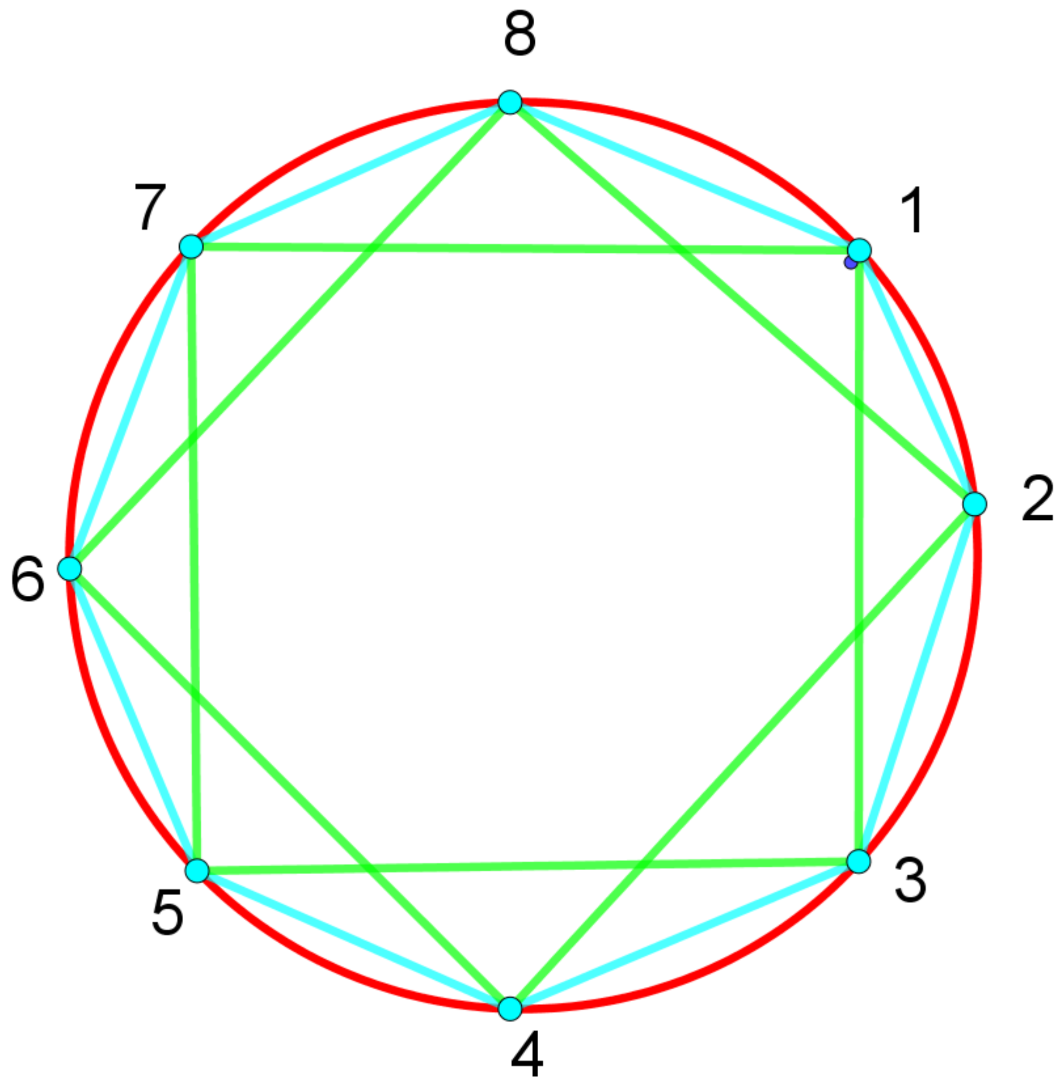
We have left out four blue points (four vertices). We can start with the first point that was left out, and go around skipping a single point. We get a second square, and the two (yellow) squares form a beautiful star:



Here the star has turned green, and we have taken it off of the original beautiful pot. The star has its own beauty:

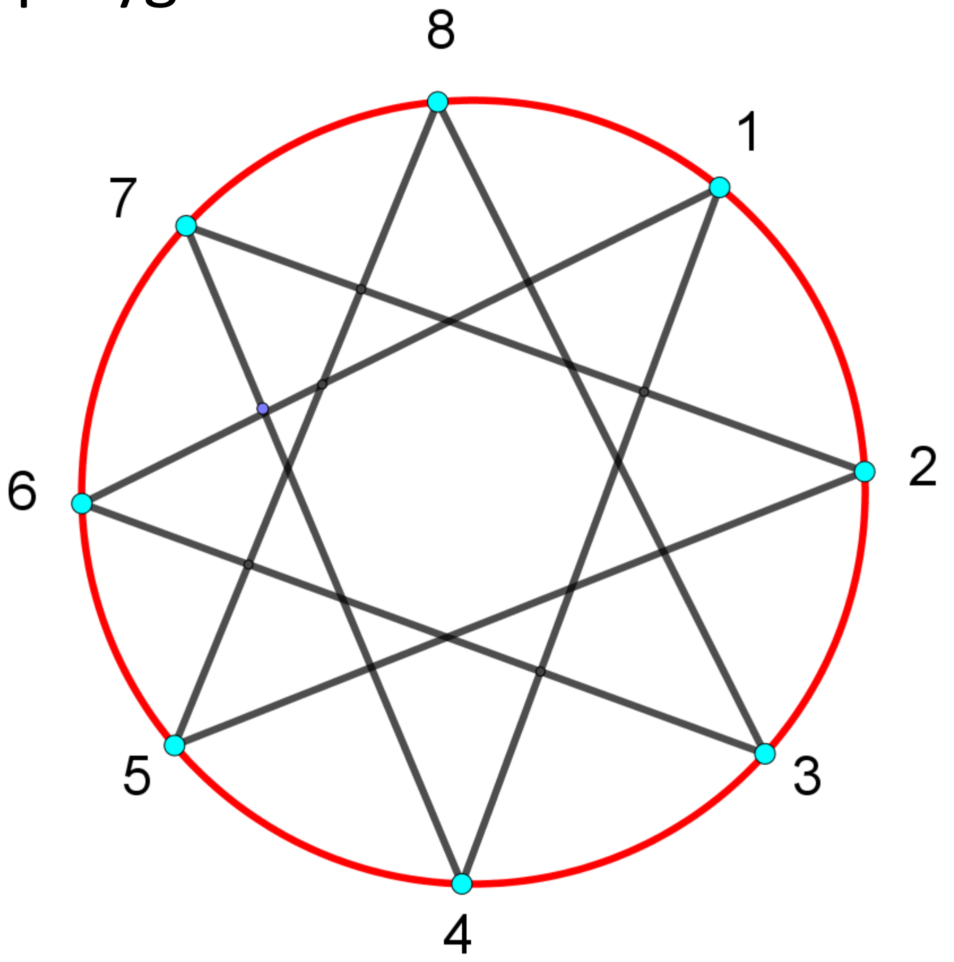


The blue figure is a polygon with 8 sides, sometimes called an ‘octagon’.
The green figure (the star) is sometimes called a ‘star polygon’.



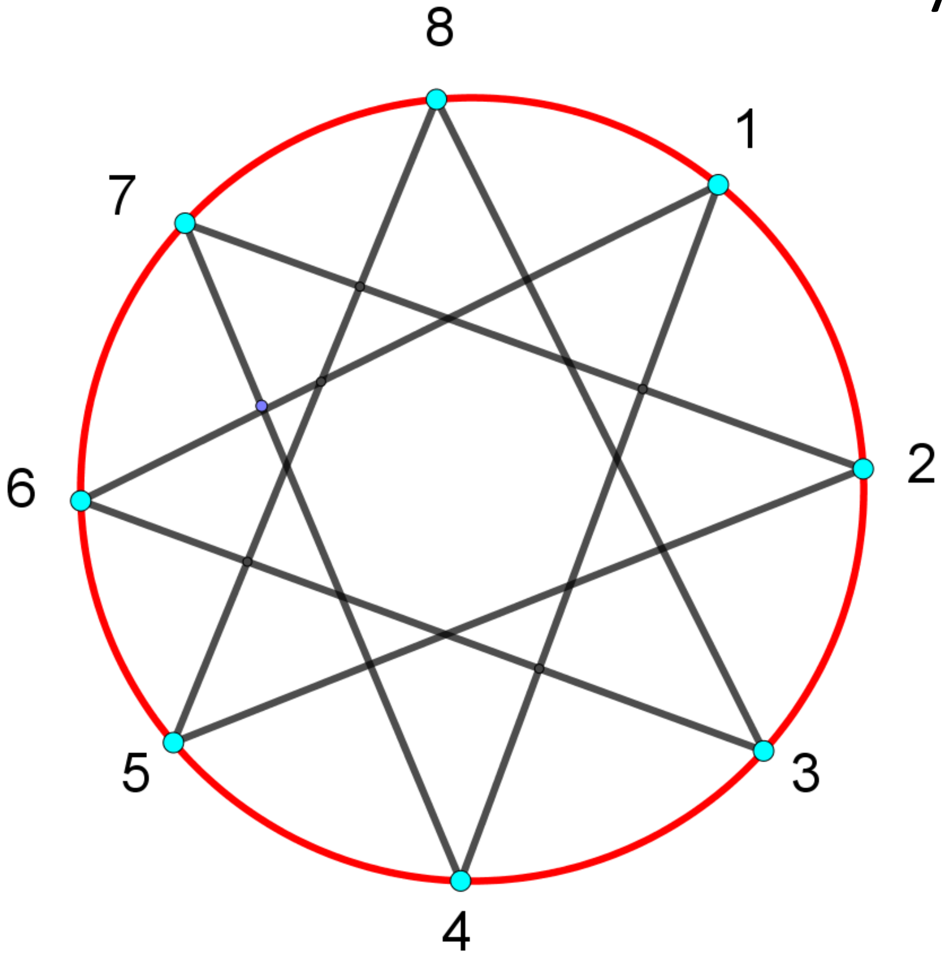
What if we start at 8 and connected every third vertex?

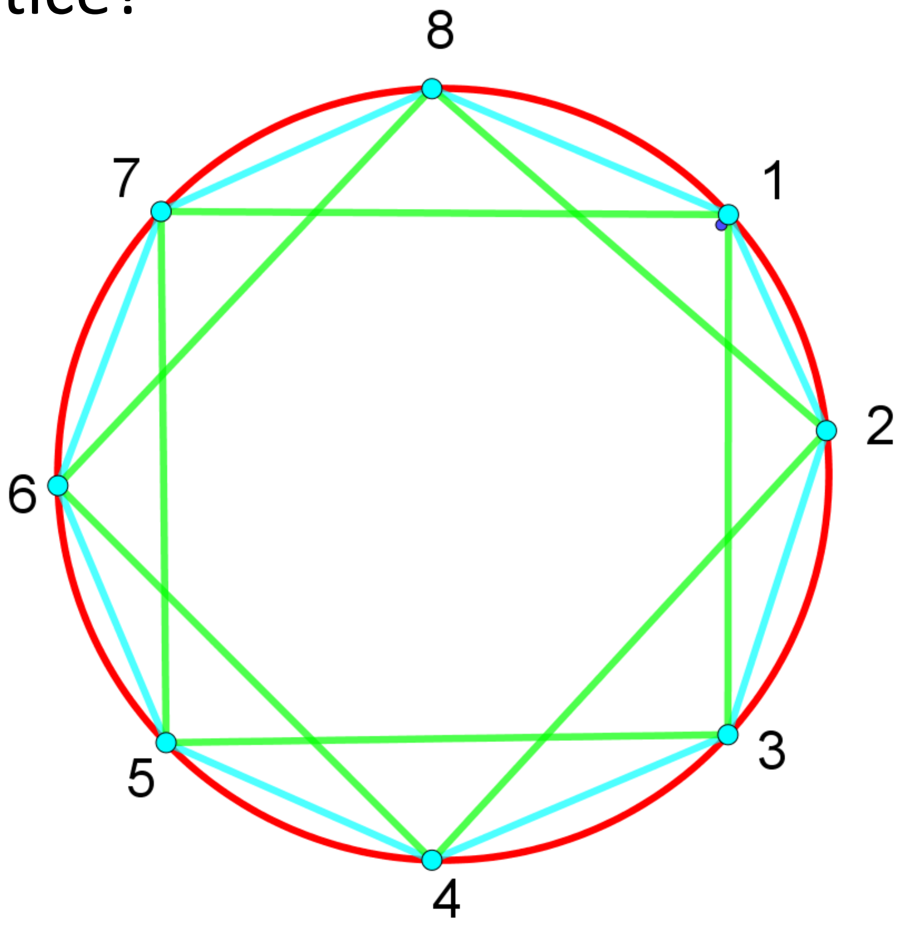
We get this new star polygon:



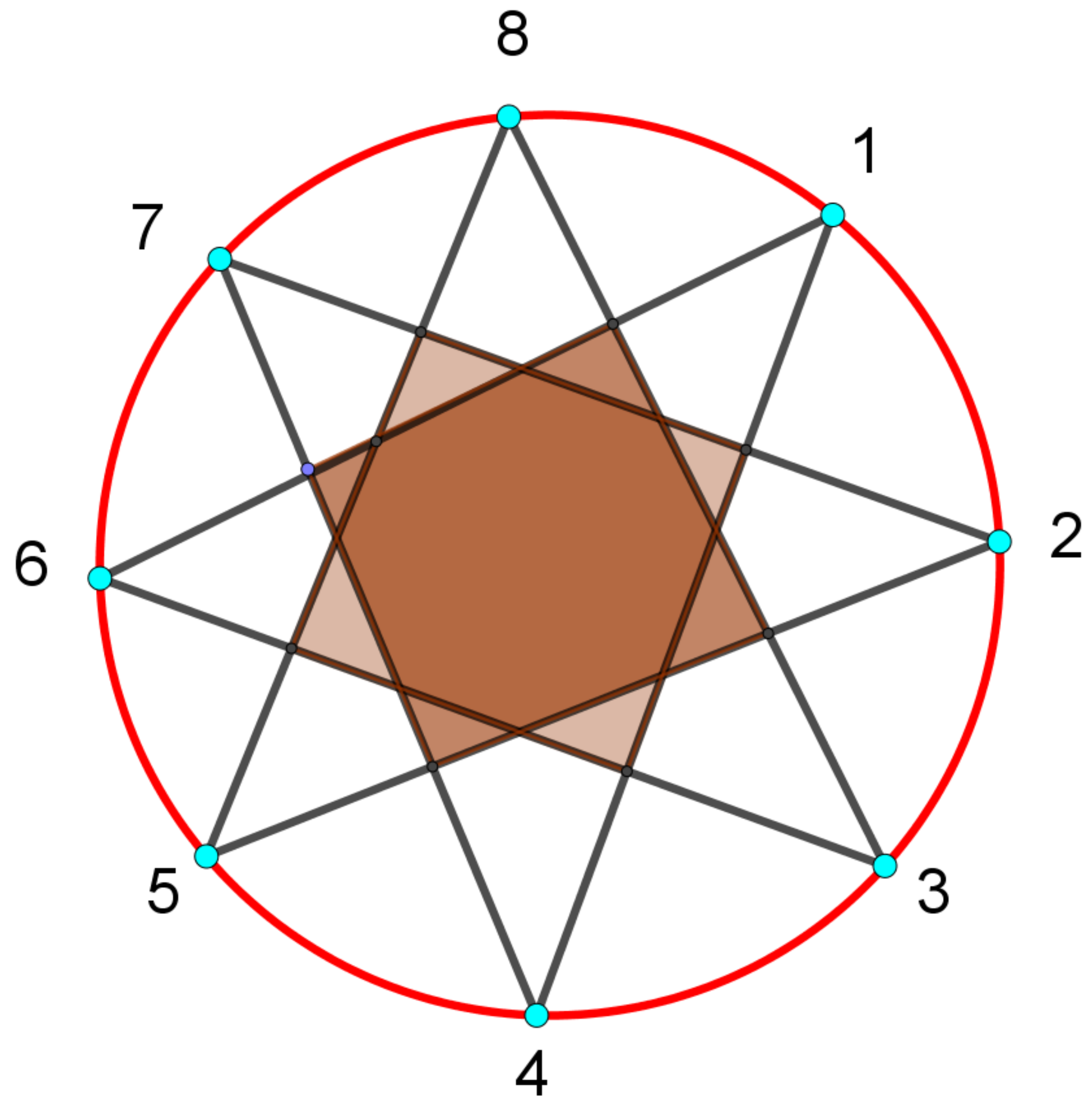
This new star is different from the star polygon formed by two squares, and also from the octagon.

What are some of the differences you notice?

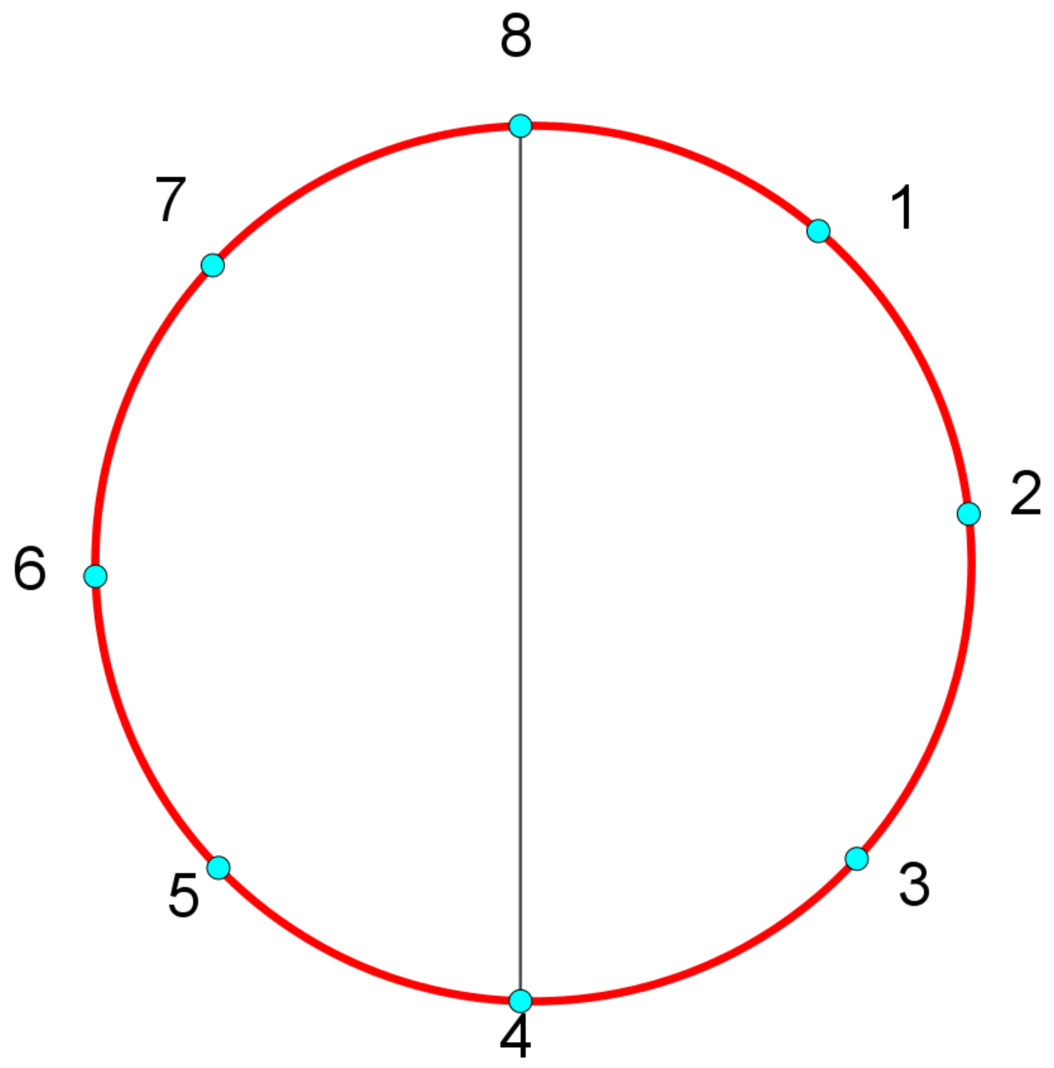




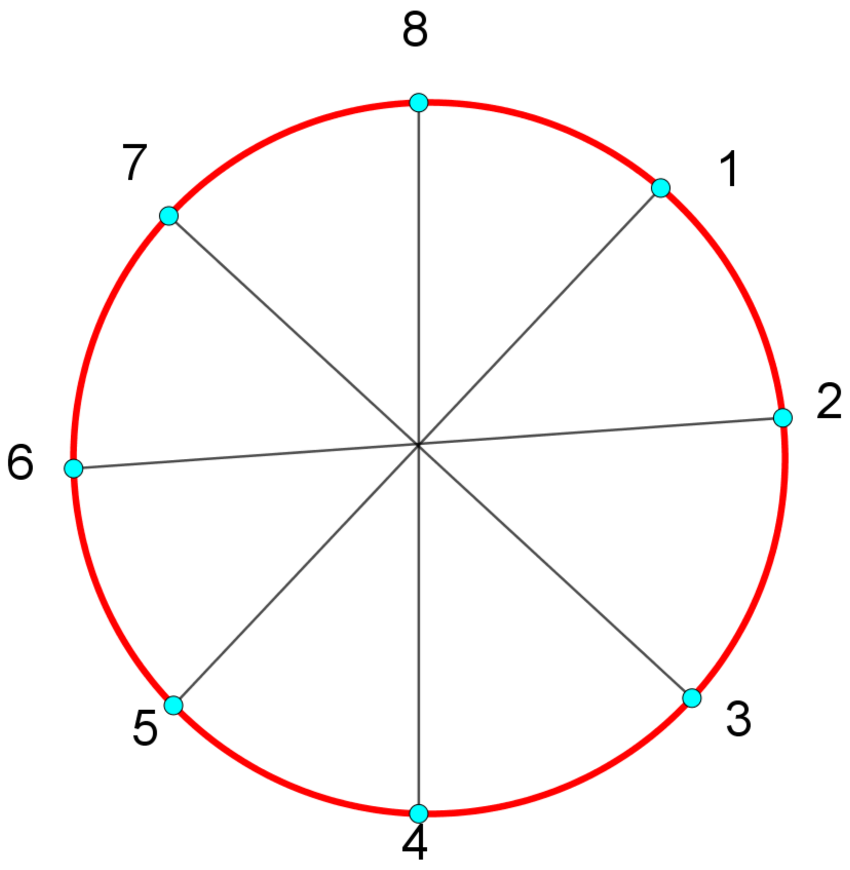
Can you see the 'double square' star inside this new star polygon?
This new star even includes the original octagon:



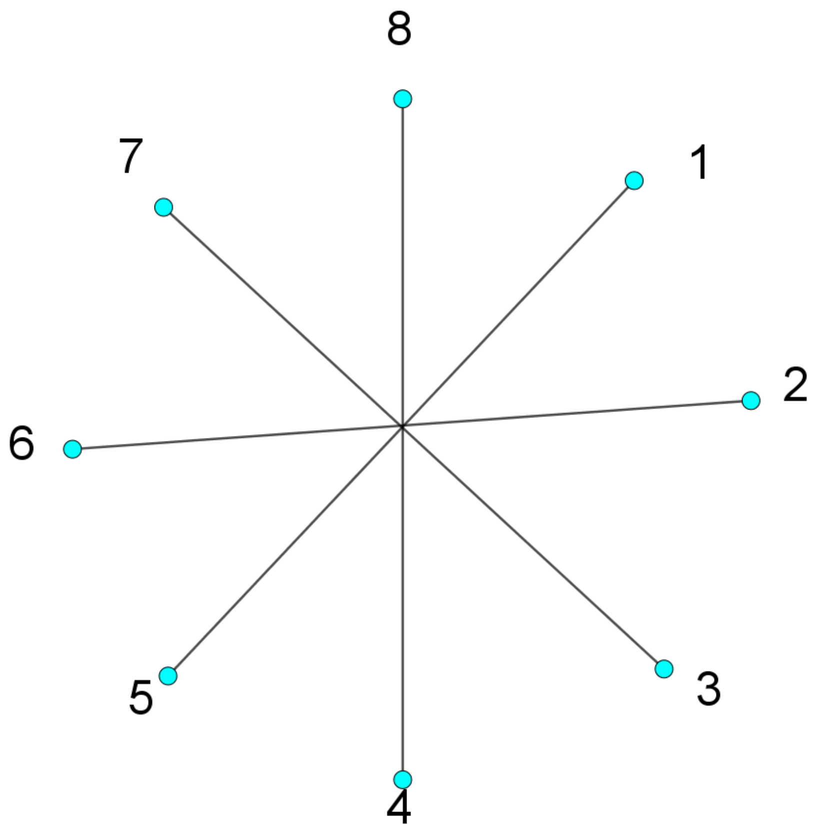
So we have connected the points of the octagon skipping no vertices, skipping one vertex, and skipping two vertices.   If we skip three vertices, we just get the diameter from vertex 8 to vertex 4.



But we leave out six other vertices. If we start with each and connect them in the same way, we get a 'skinny' star (an 'asterisk'):



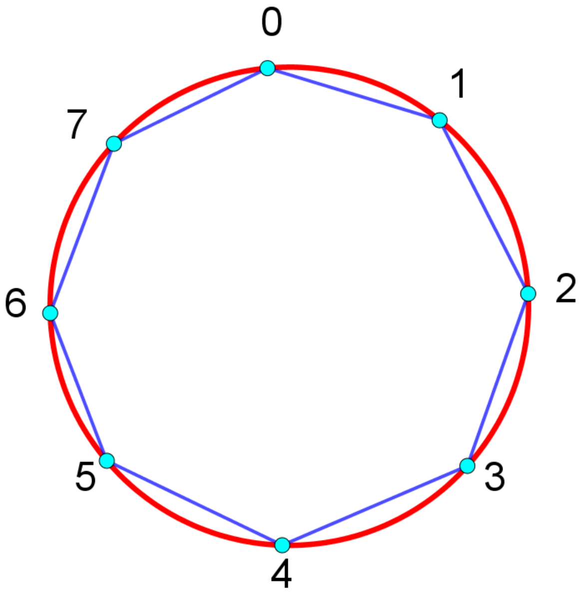
(with its circle)

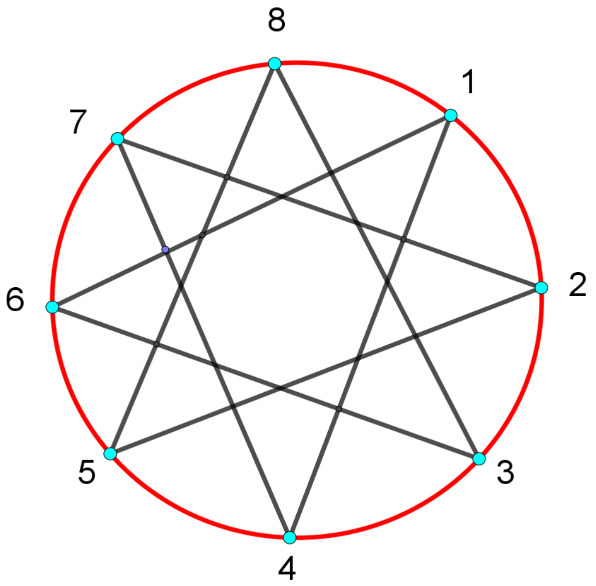


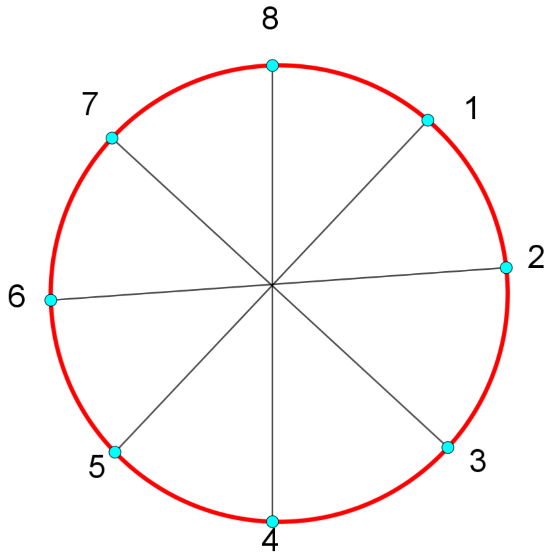
(without its circle)

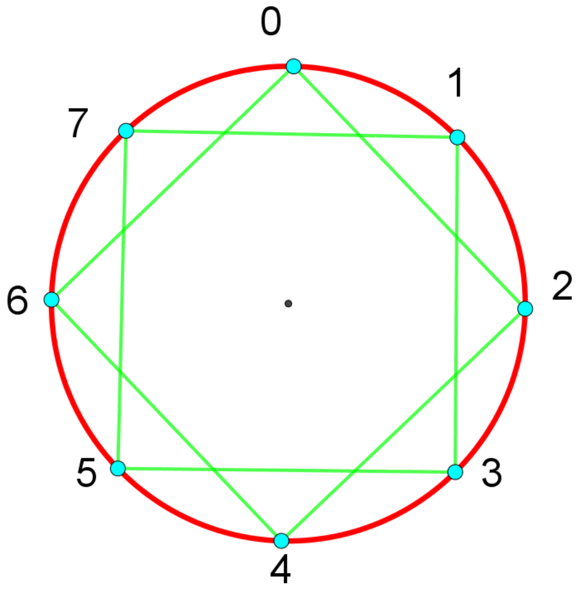
Notice that both the octagon and the star polygon skipping two points can be drawn as one continuous line.

But the ‘two squares’ star and the asterisk require more than one continuous line.

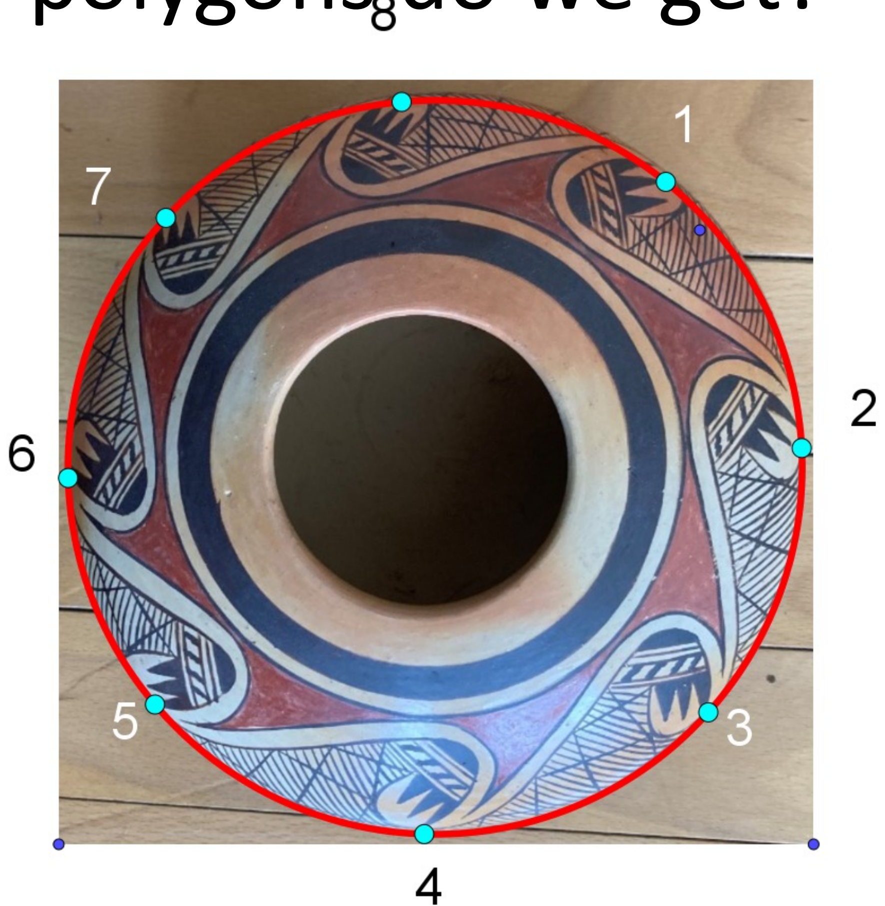




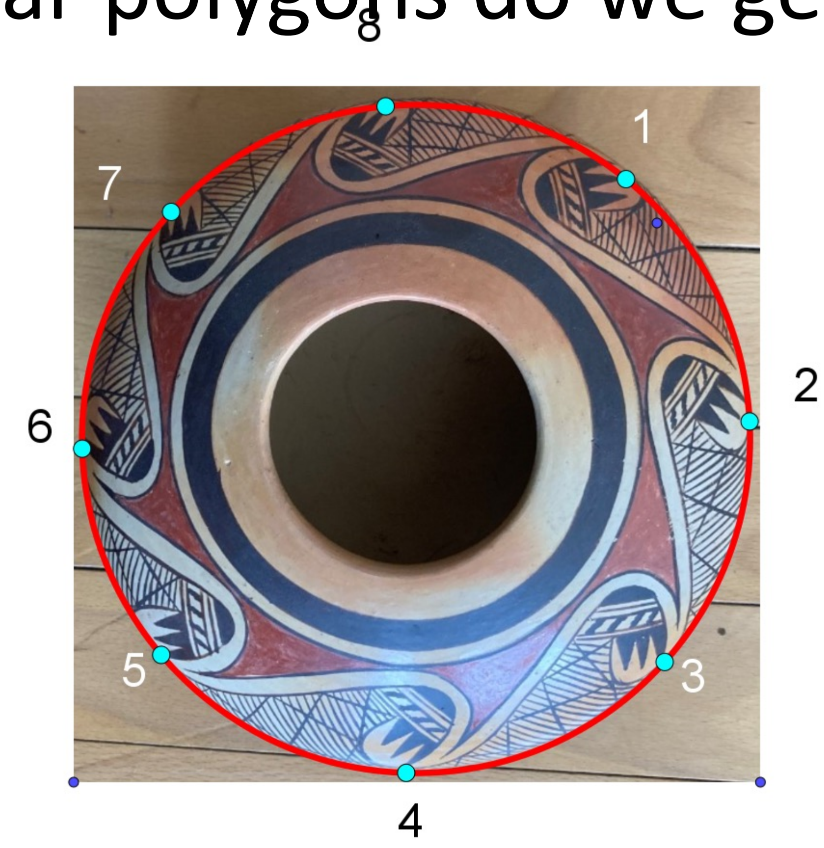




Counting the octagon and the asterisk as a star, we now have four kinds of star polygons from the Nampeyo pot. What if we start at vertex 8 and skip 4, 5, 6, or 7 vertices? What kinds of star polygons do we get?

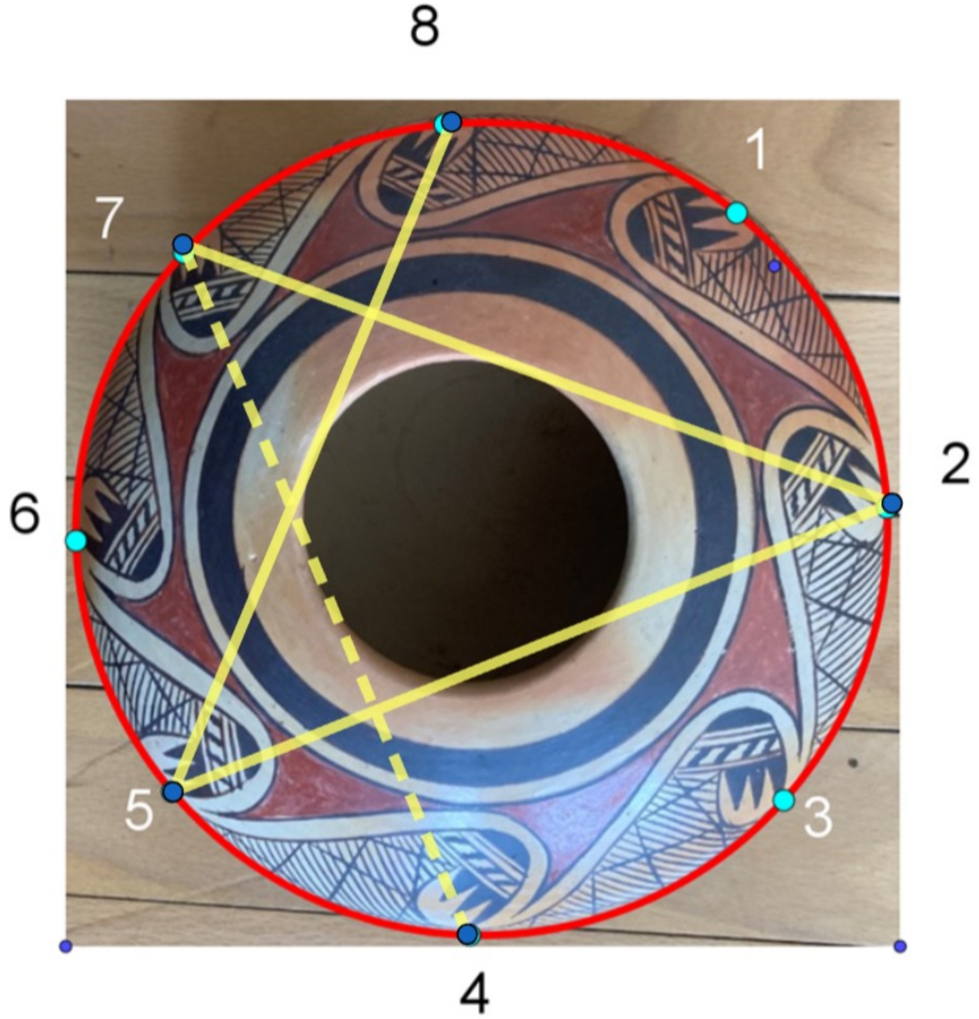


Counting the octagon and the asterisk as a star, we now have four kinds of star polygons from the Nampeyo pot. What if we start at vertex 8 and skip 4, 5, 6, or 7 vertices? What kinds of star polygons do we get?



We get the same stars, traced 'backwards'!

The diagram below shows how we go from point 8 to point 5 clockwise(skipping 4 points). It's the same as skipping just 2 points, but going counter-clockwise.



Here is a pot by Anderson Peynetsa, a potter from the village of Zuni in New Mexico:

The Zuni are another group of native Americans who have lived in the southwest for thousands of years.

They live in the state of New Mexico.

(New Mexico is the name of one of the states of the United States. It is not in the country of Mexico.)

Like the Hopi, the Zuni were traditionally agriculturalists (farmers). But they speak a completely different language and hold different religious beliefs. And their pottery is also different:

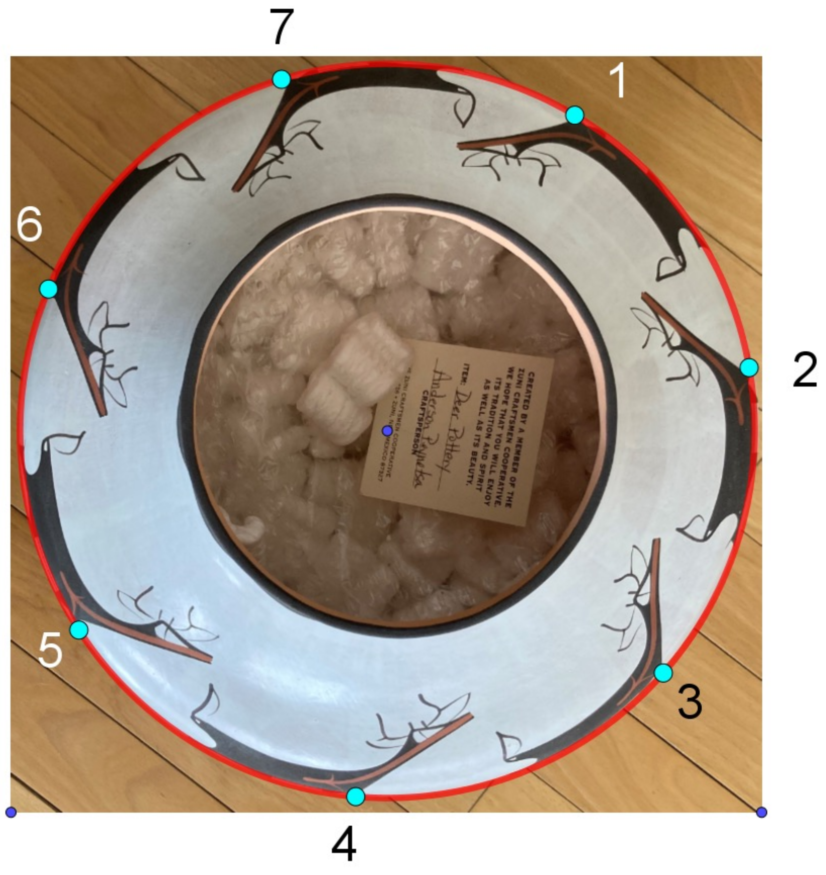
Zuni Elementary School

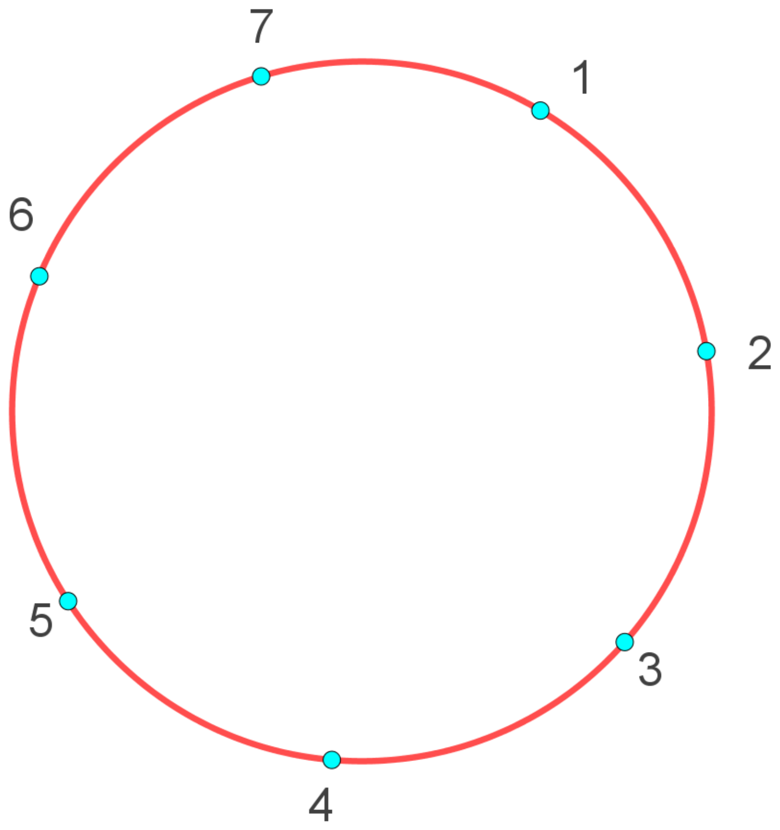
Here is the top view of Peynitsa's bowl. He has put seven deer figures around a circle.

Let us make star polygons with seven points, starting with Peynitsa's bowl.

Can we make a regular polygon (regular 7-gon)?

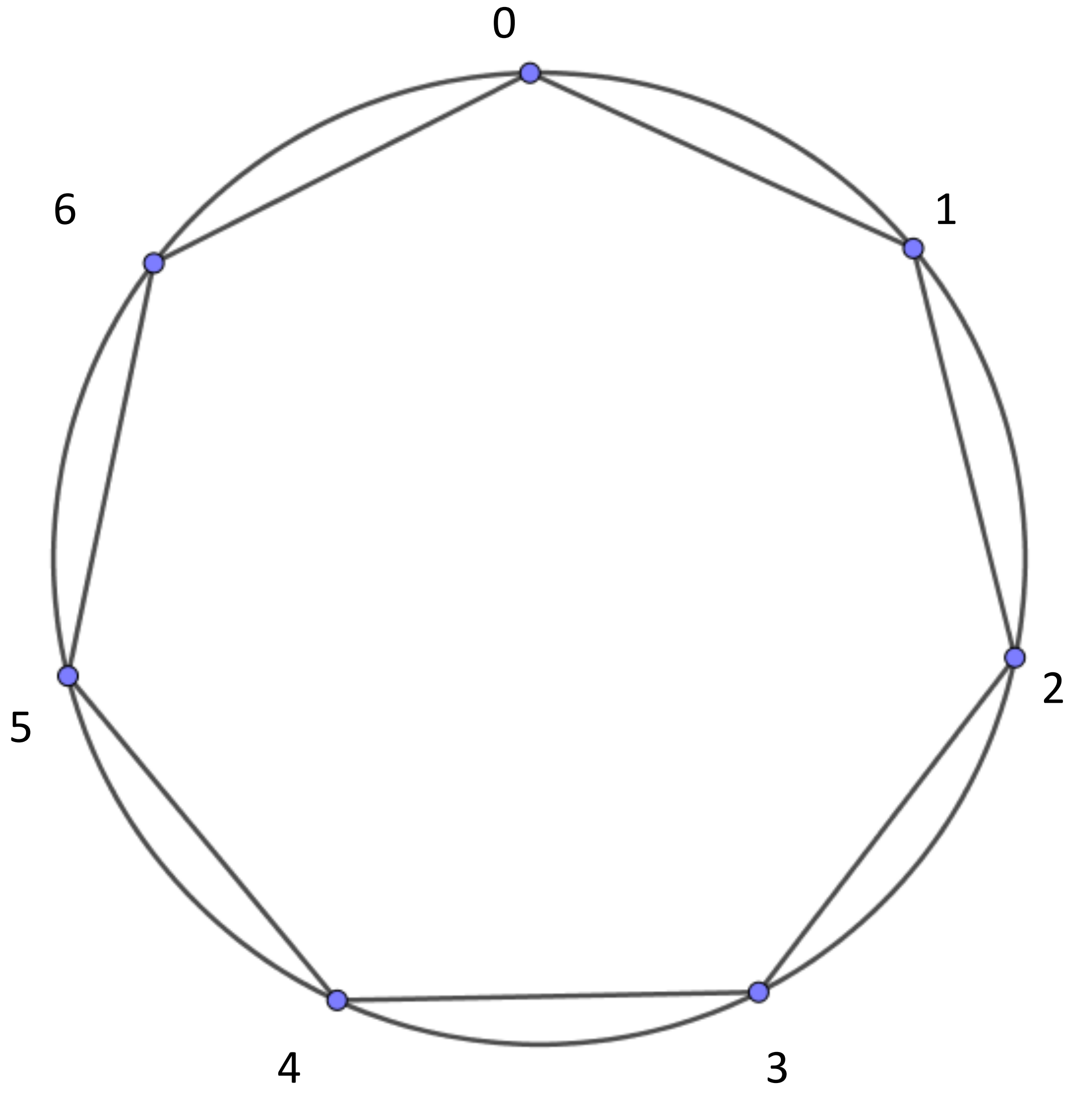
Can we make an asterisk?





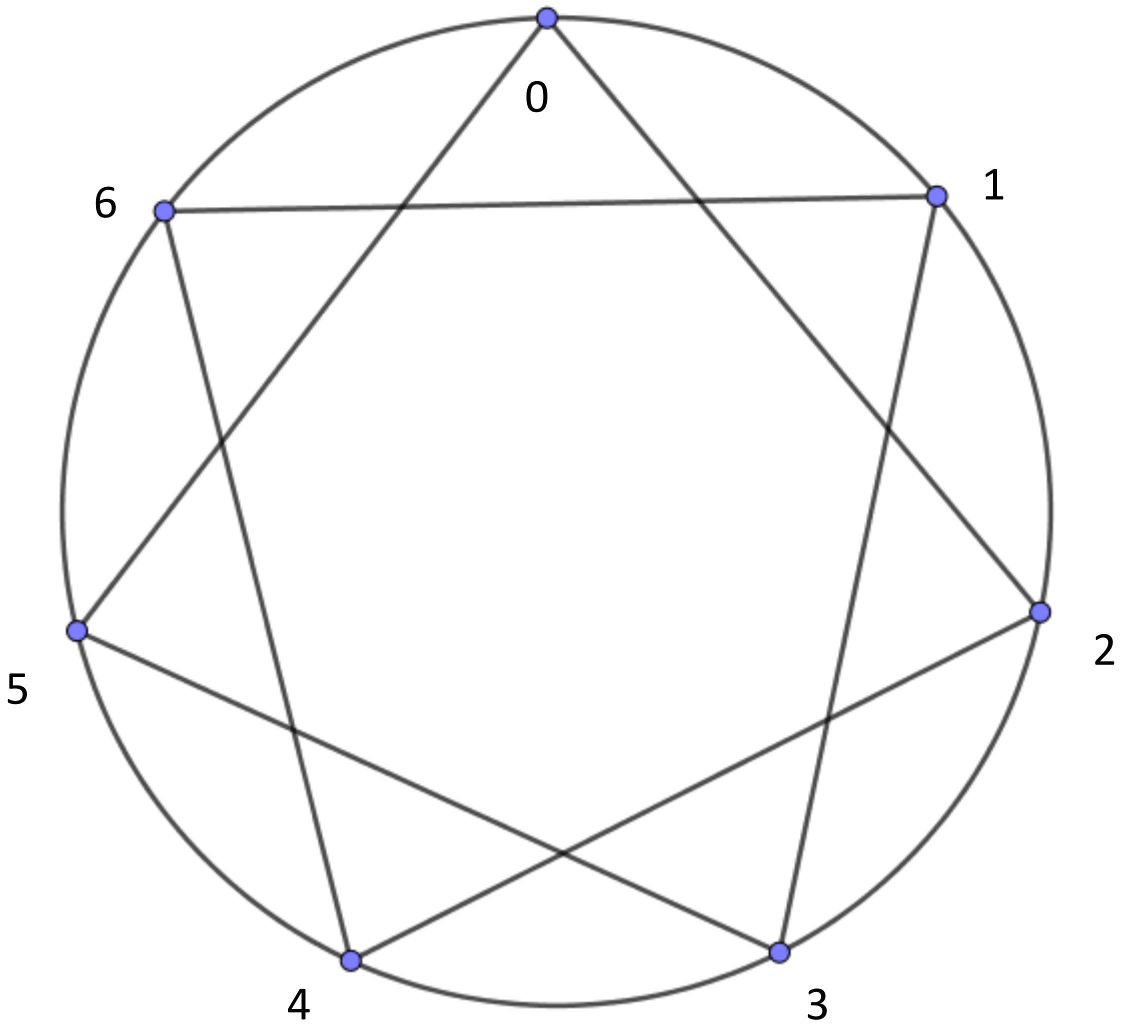
Here is a regular 7-gon drawn on Peynitsa’s bowl:

Here is a regular 7-gon drawn without the bowl. We will number the vertices from 0 to 6 (rather than from 1 to 7, just because…)

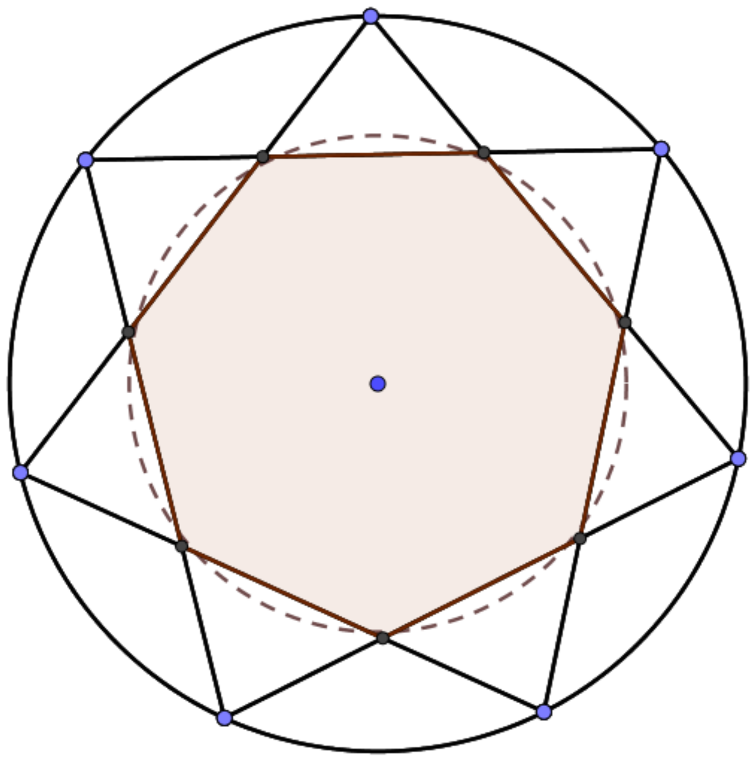


Here we draw a star polygon by skipping one point: starting from 0, we go to 2, etc.

Notice that this polygon is formed from one continuous line.

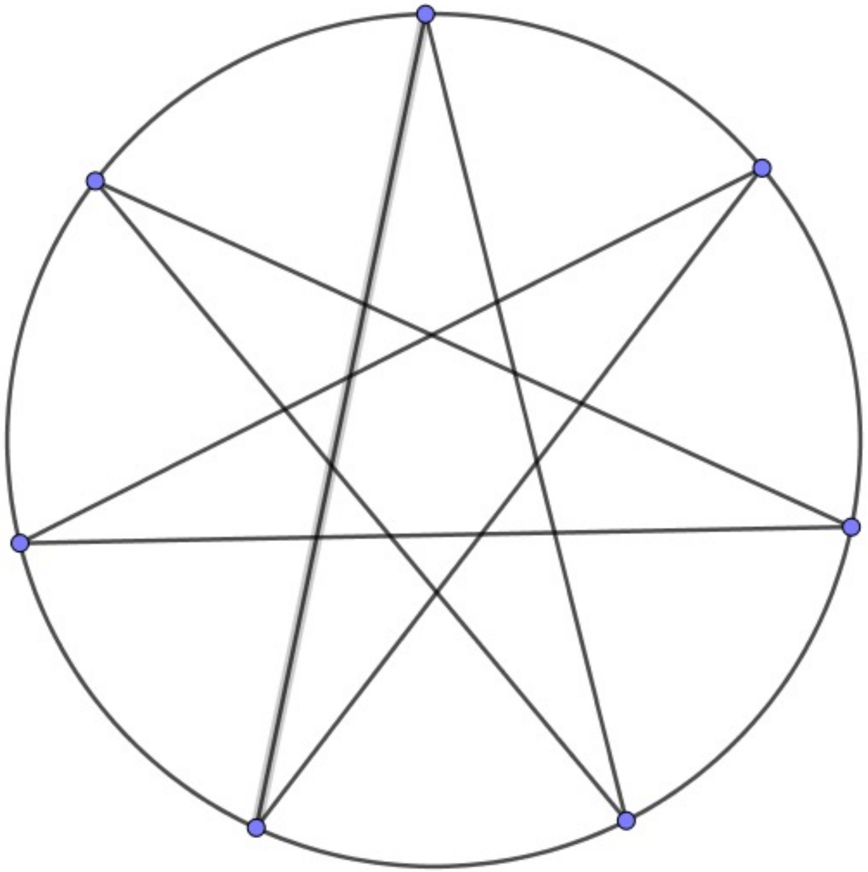


And it contains the regular 7-gon:



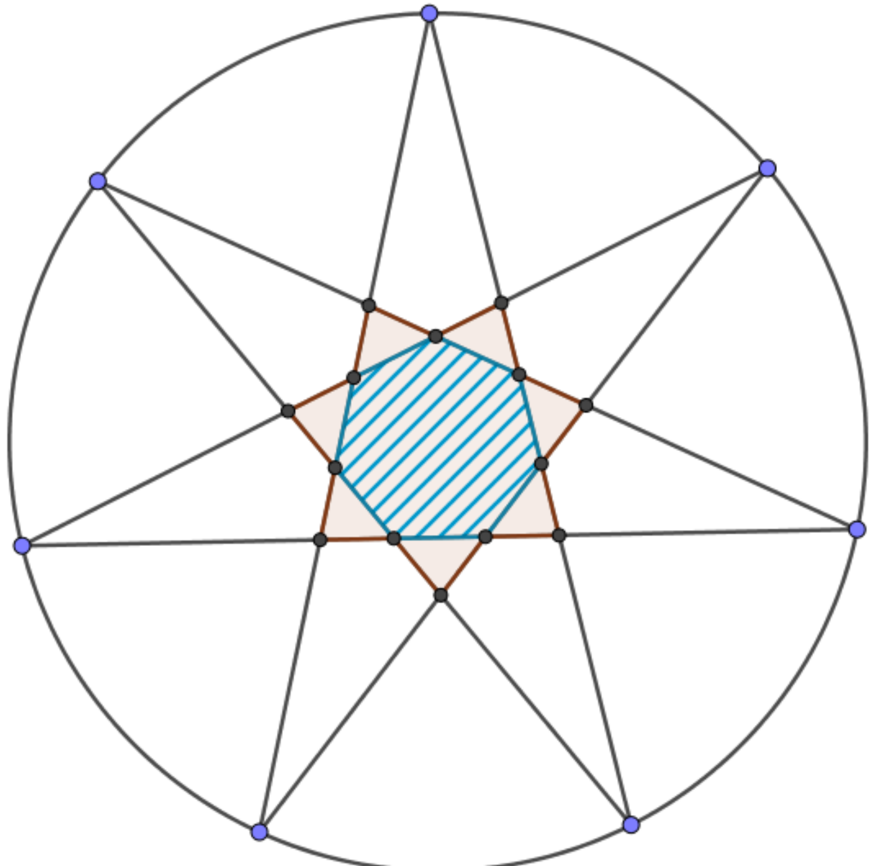
Here is the star drawn by skipping two points.

Does it consist of one continuous line?
Does it include any of the stars we've already drawn?



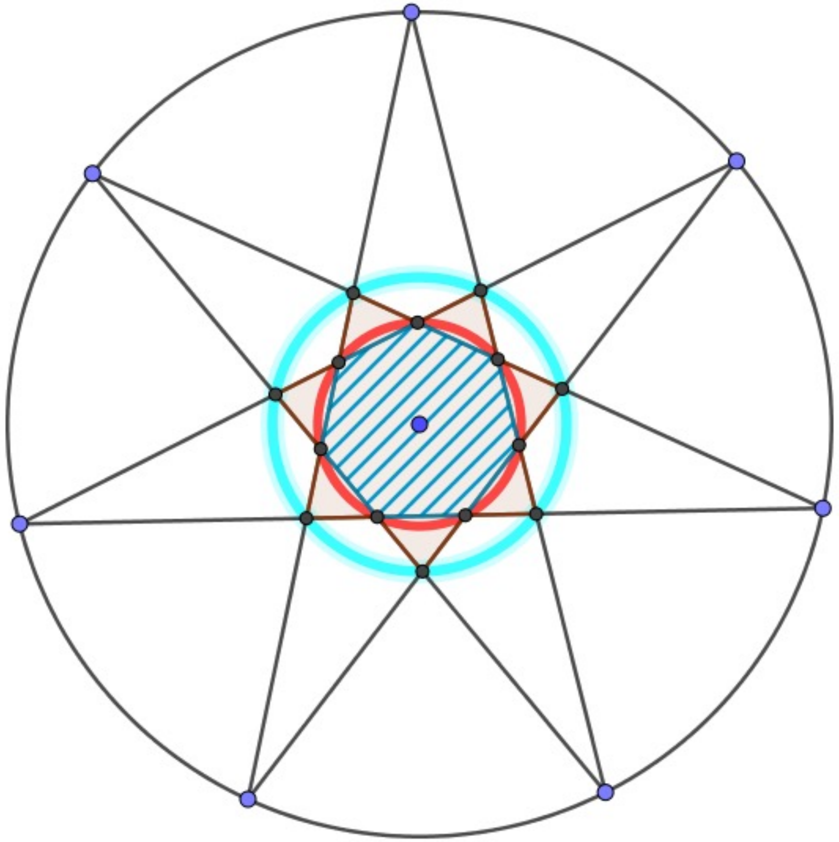
Here is the star drawn by skipping two points.

It consists of one continuous line.
It includes the 7-gon (blue stripes) as well as the star drawn by skipping one point.



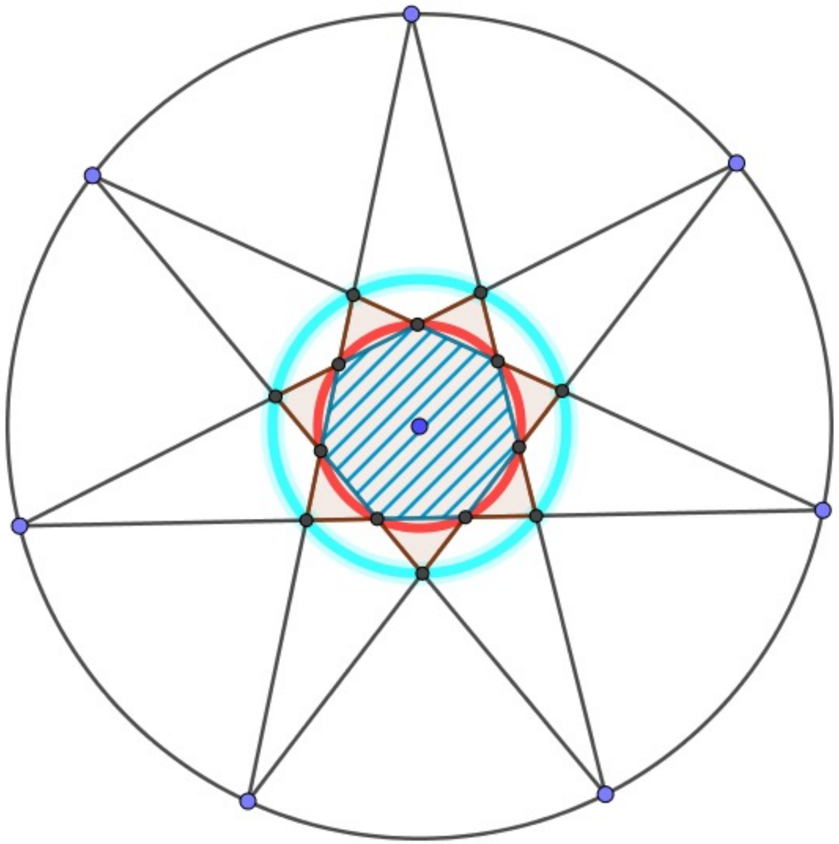
Challenge: Each of the stars contained in the skip-3 star can be measured by the radius of its circumcircle. If the smallest (red) circumcircle has radius 1 what are the radii of the other two circumcircles?

In other words, what are the ratios of their radii?



Challenge: Each of the stars contained in the skip-3 star can be measured by the radius of its circumcircle. If the smallest (red) circumcircle has radius 1 what are the radii of the other two circumcircles?

In other words, what are the ratios of their radii?



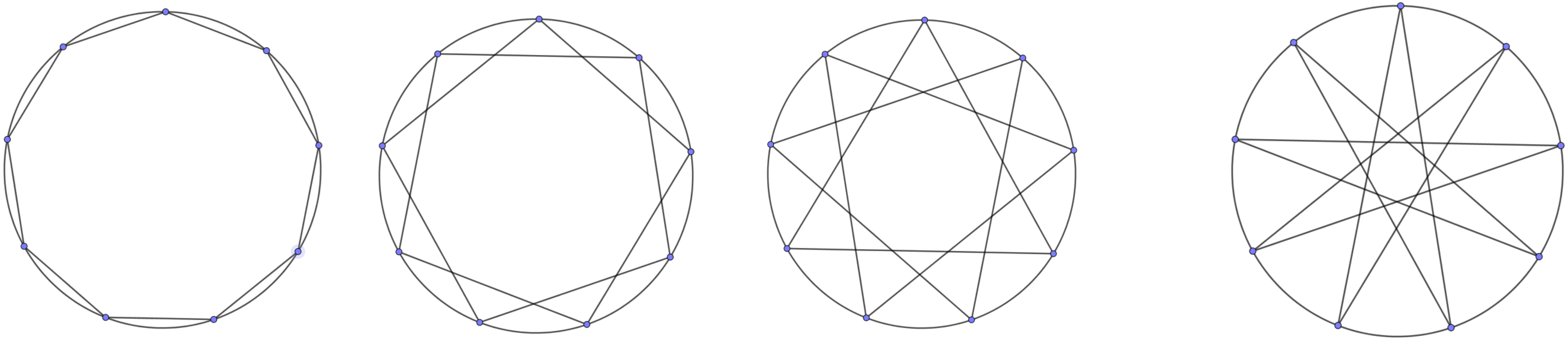
That is homework for you.

I'll check back next year to make sure you've done it.

Here are the four possible nine-point star polygons.

How was each formed?

How does each confirm or disprove your hypotheses?



CHALLENGE: Choose a number N of equally spaced points around a circle.

Draw as many different stars as you can.

Can you ‘predict’ which stars are formed by continuous lines? And which stars will ‘close’ before they include every vertex?

CHALLENGE: Choose a number N of equally spaced points around a circle.

Draw as many different stars as you can.

Can you ‘predict’ which stars are formed by continuous lines? And which stars will ‘close’ before they include every vertex?

For example, if you had 24 points around a circle, would the star connecting every third point consist of a single line? Or will you have to start over in drawing it?

What about the star connecting every fifth point of the 24 points?

CHALLENGE: Choose a number N of equally spaced points around a circle.

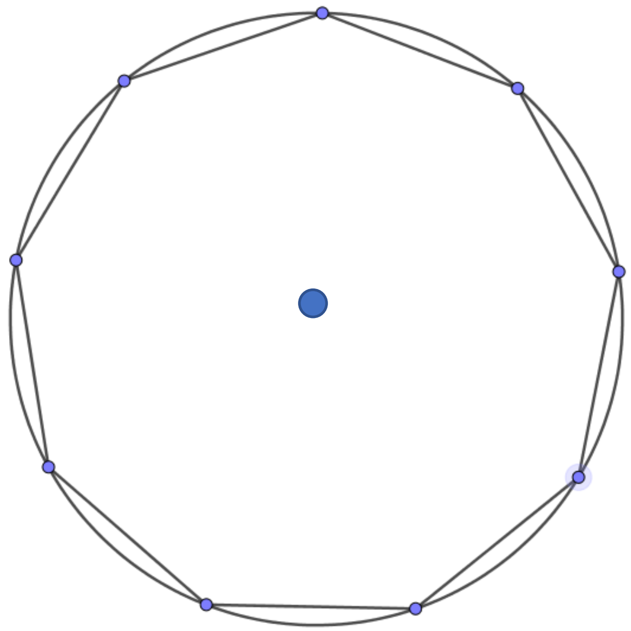
Draw as many different stars as you can.

Can you ‘predict’ which stars are formed by continuous lines? And which stars will ‘close’ before they include every vertex?

Or, if you have 24 points around a circle, how many stars will consist of a single line? How many will require starting a new line?

What about 34 points? Can you generalize your observations?

Here is another interesting way of looking at these star polygons.

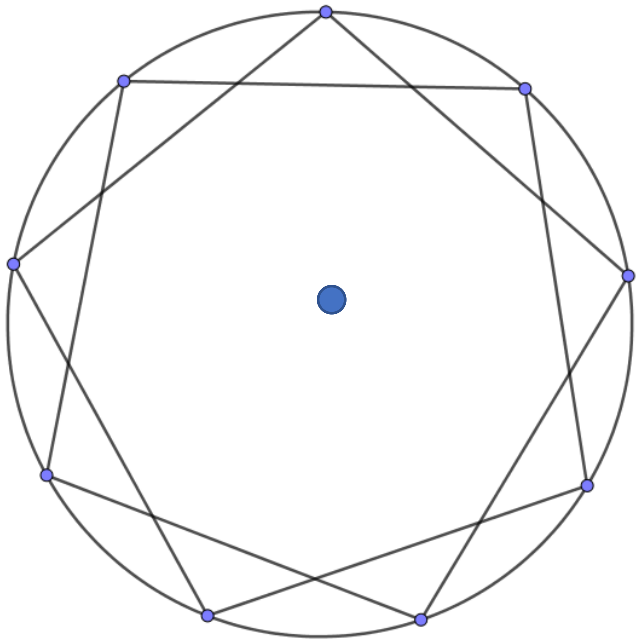


Start at any vertex of this simple star (which is just a regular 9-gon), and 'walk' around it. Think of the center of the circle as a pole, and think of holding a ribbon attached to the pole.

As you walk around the polygon, the ribbon winds around the pole.

When you have returned to the vertex you started from, the ribbon will have wound once around the pole.

Here is another interesting way of looking at these stars.



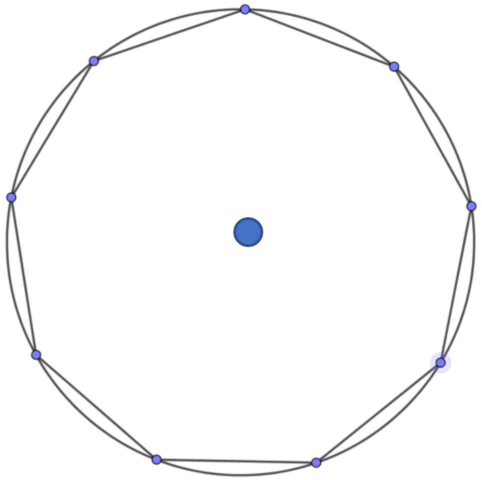
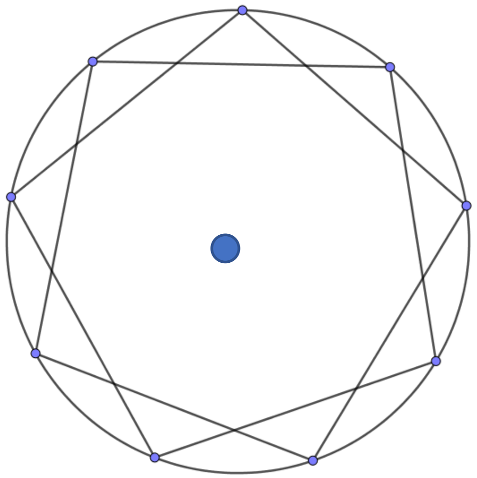
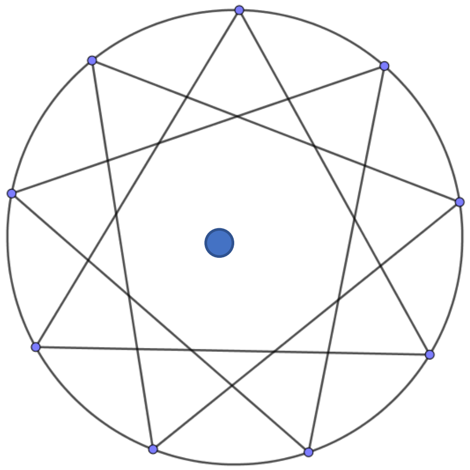
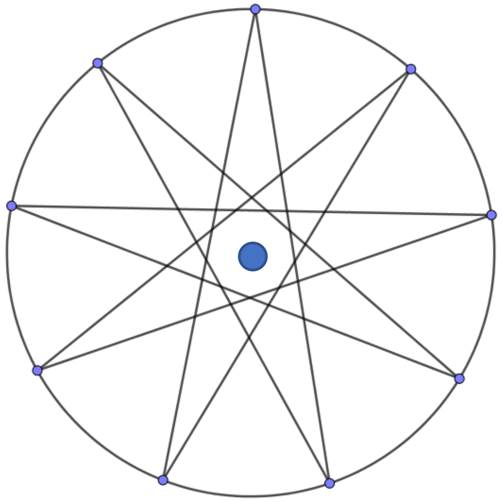
Now start at a vertex of this star polygon, and walk around until you come back to the vertex.

This time, the ribbon will wind twice around the pole.

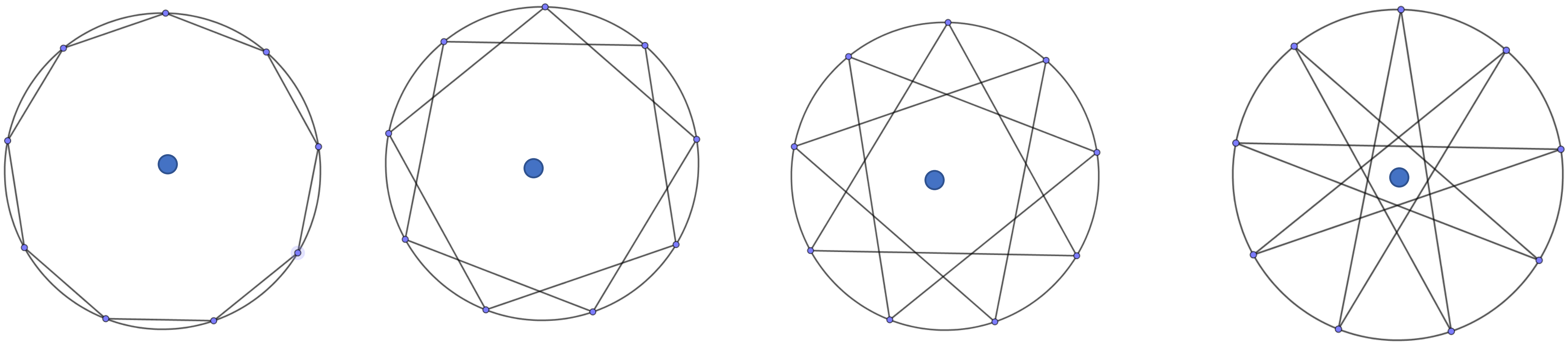
We say that the ‘winding number’ or ‘density’ of this star polygon is 2.

The winding number of the simple 9-gon is just 1.

What is the winding number of each of these nine-point star polygons?

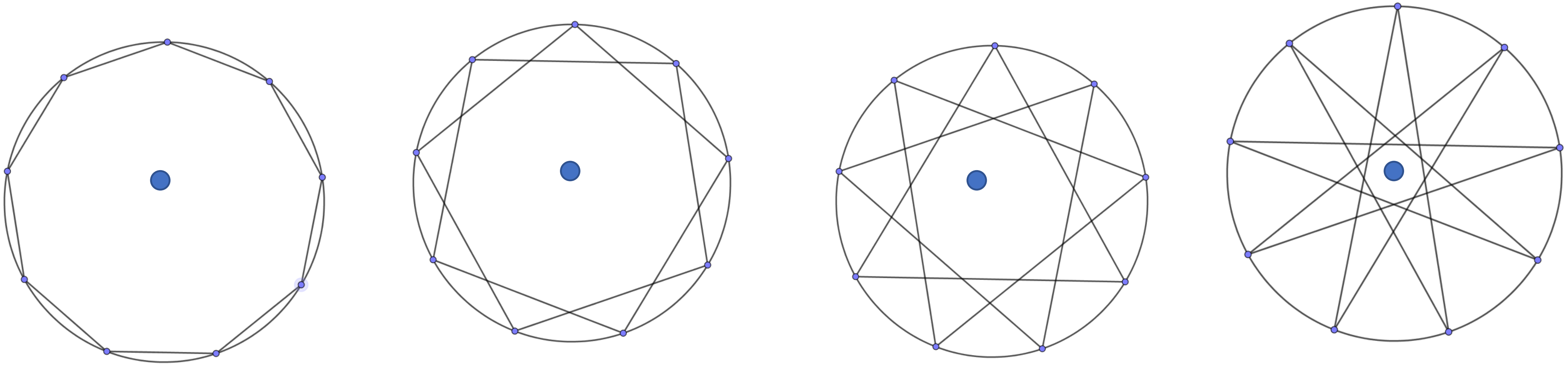





What is the winding number of each of these nine-point star polygons?



1 2 UMM… 4

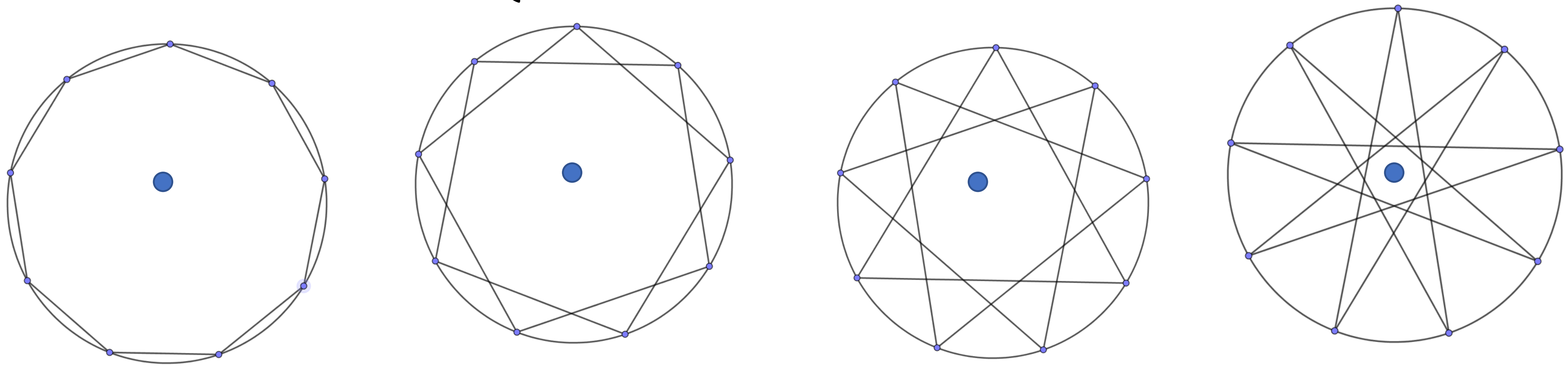
What is the winding number of each of these star polygons?



SOME QUESTIONS TO THINK ABOUT

1) How is the winding number related to the number of points skipped? To the number of the first vertex connected to the 0 vertex?

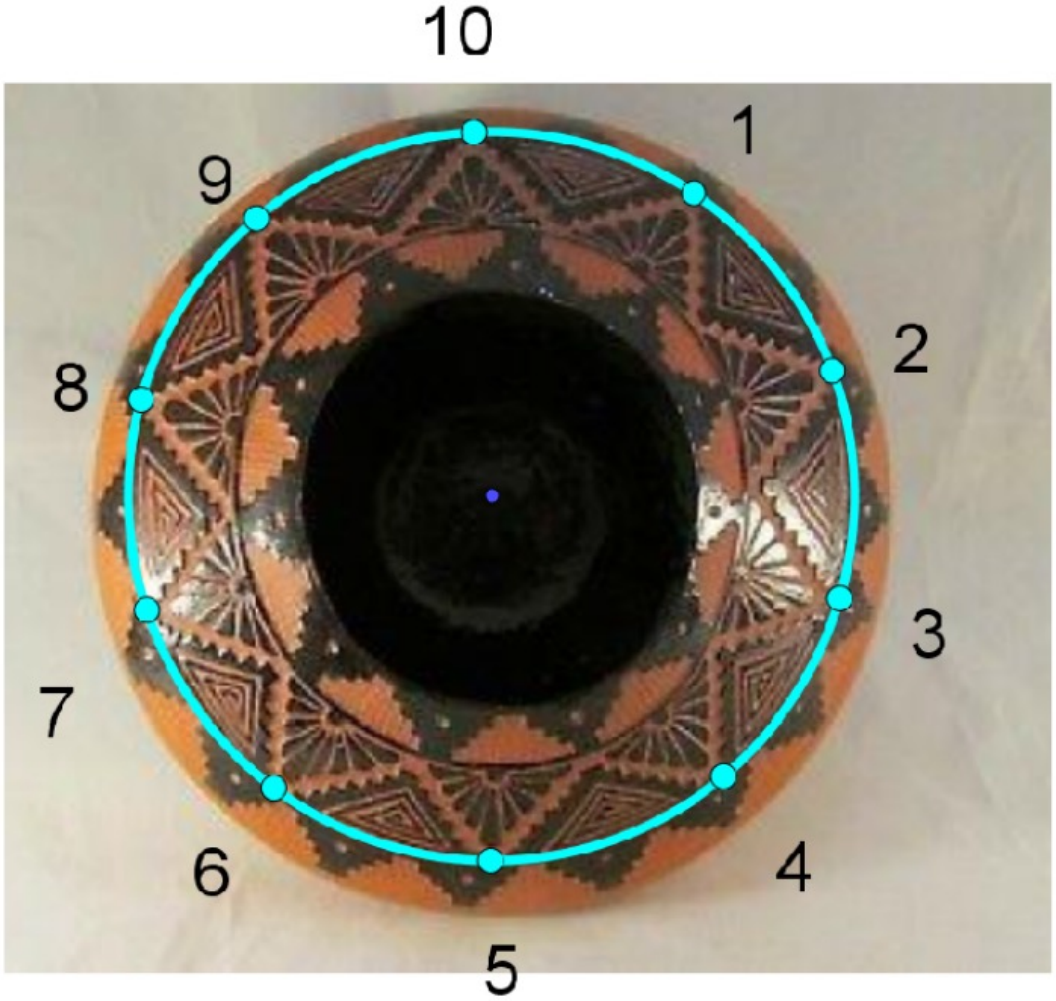
SOME QUESTIONS TO THINK ABOUT



2) (If you know some geometry…) How is the winding number related to the size of the angle at one of the vertices of each star polygon?

3) What might it mean to have a star polygon with a winding number of -2? (That’s “negative 2”.)

This is a bowl by a Navajo artist.



The signature on this pot is given below.

The Navajo people also live in the states of Arizona and New Mexico.

They settled there later than the Hopi or the Zuni. They moved down from the north about 600 years ago.

Unlike the Hopi or the Zuni, the Navajo traditions were based on hunting (and later sheep herding), rather than farming.

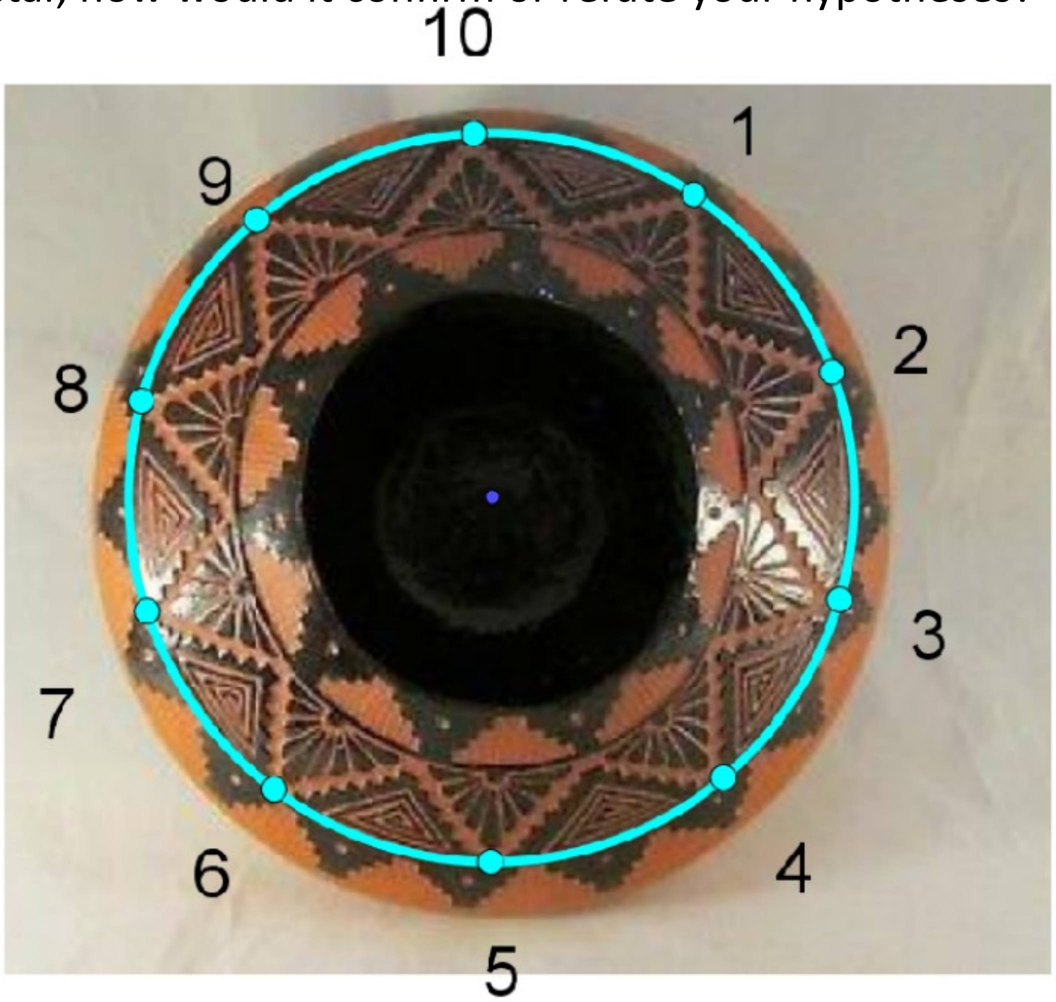
Their language is unrelated to Hopi or Zuni. It is related to *Athabascan* languages, spoken mostly in Canada.

This is Diné college, on the Navajo reservation.

Here is that bowl by a Navajo artist. The artist has drawn a 10-point star around the outer circle, here show in light blue.

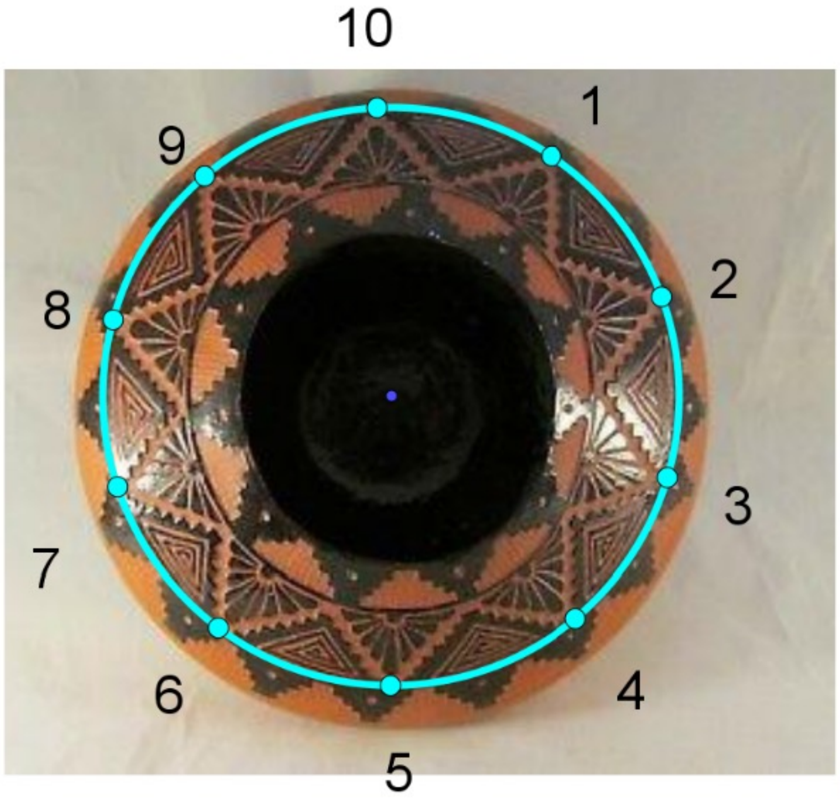
How did the artist draw this star?

If you drew the whole star, how would it confirm or refute your hypotheses?



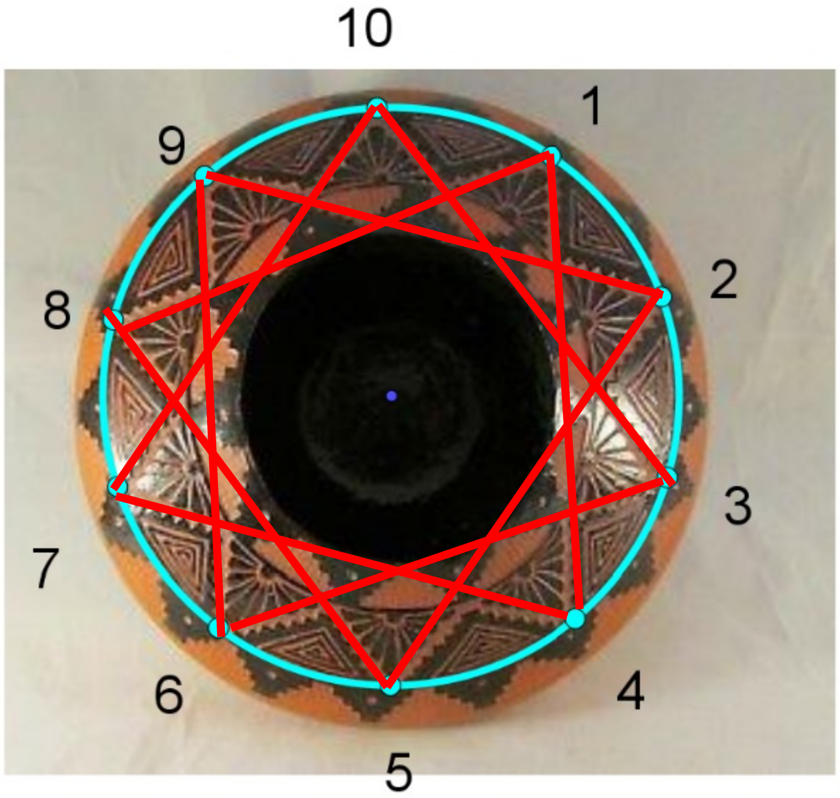
The signature on this pot is given below.

In this Navajo pot, the artist has already drawn a 10-pointed star. But he or she has left only the points of the star. We can draw in the rest of the star polygon.



The surface of the pot is curved, so that the lines we draw may not be right on the lines in the picture.

We can draw in the rest of the star polygon:

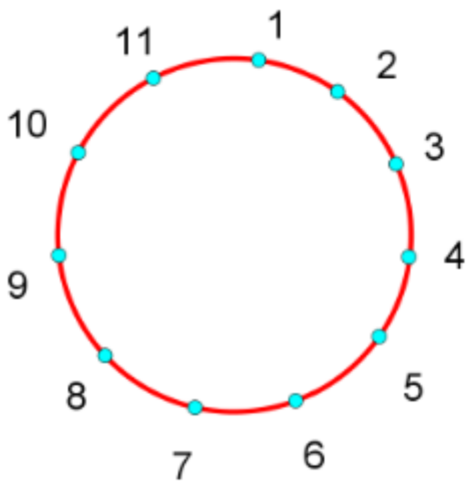


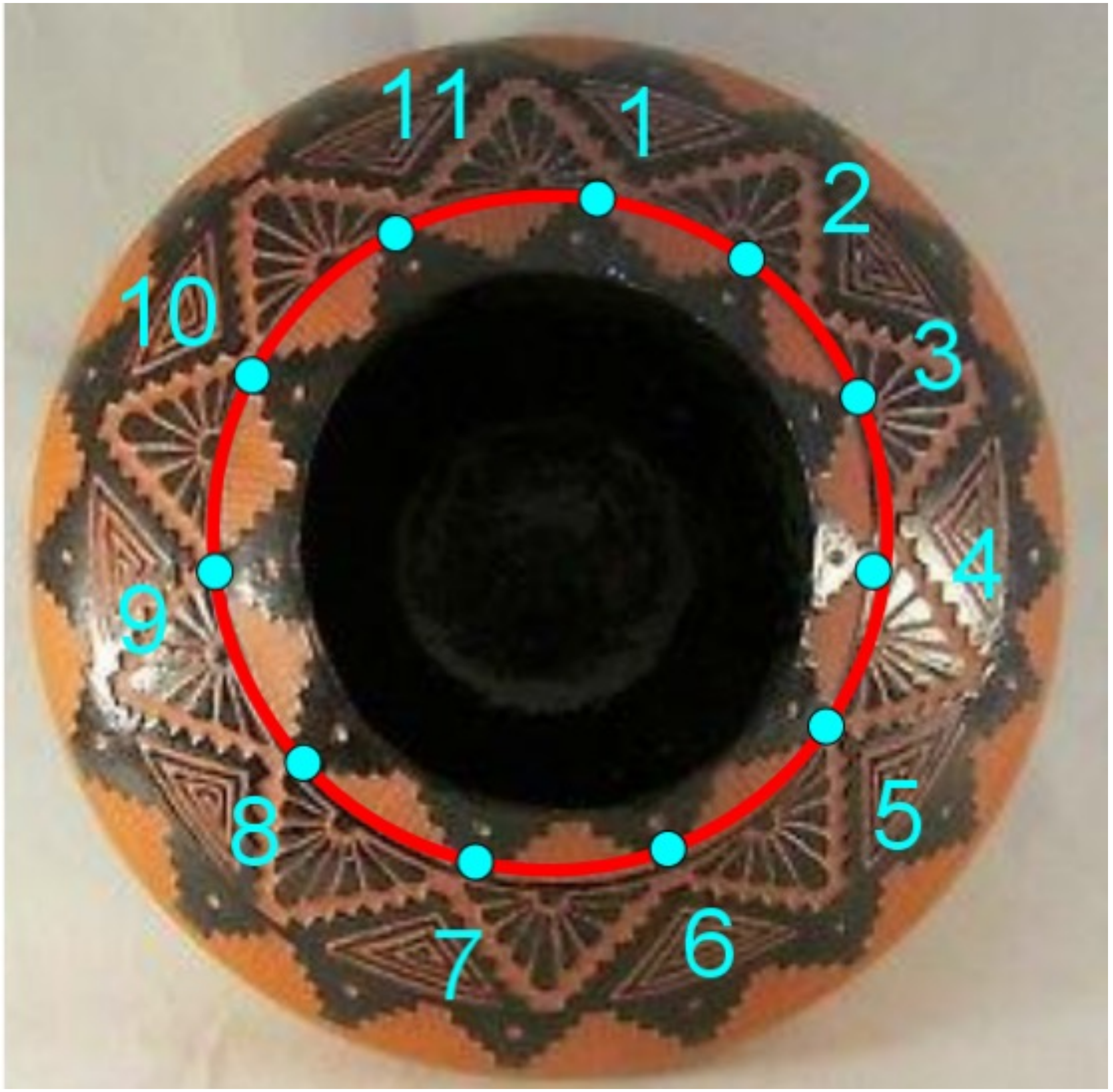
The surface of the pot is curved, so that the lines we've drawn may not be right on the lines in the picture.

If you look closely at this remarkable pot, you will see that the artist created an 11-pointed star as well as a 10-pointed star.

Again, she or he has shown just the points of the star.

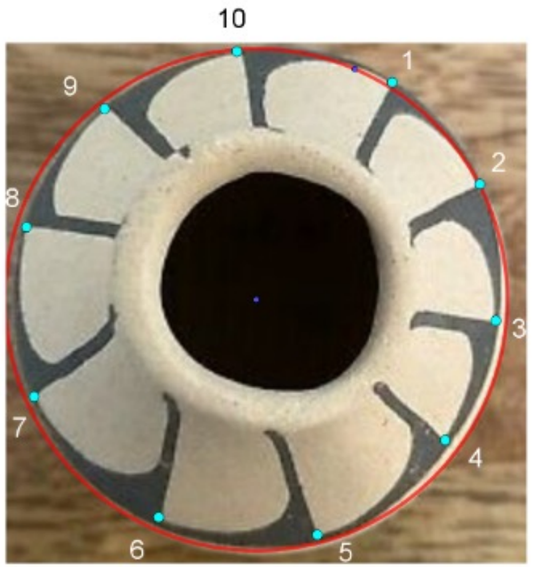
Can you trace this rest of this 11-point star? It is created by skipping two points at a time around the circle.

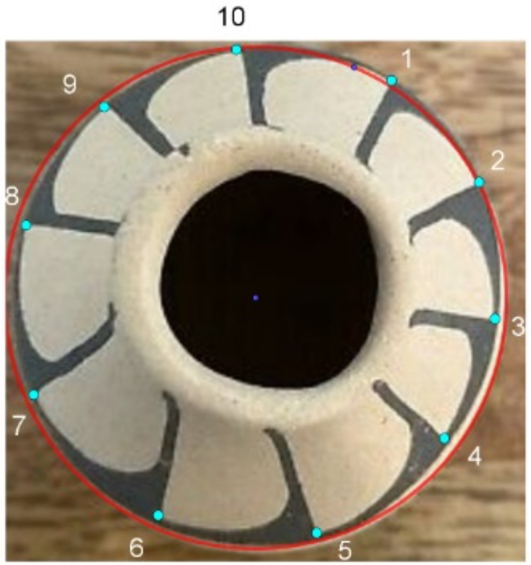


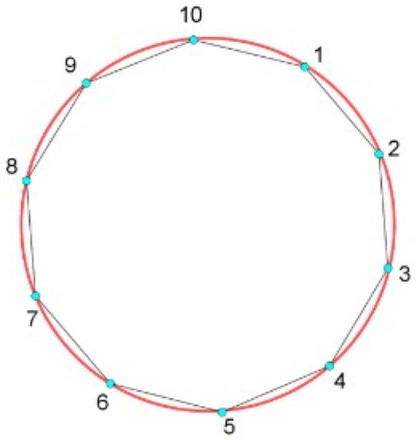


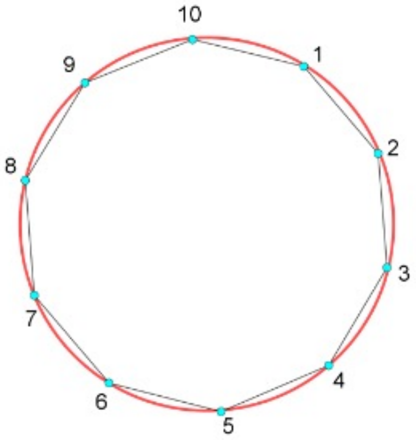
Try for yourself:

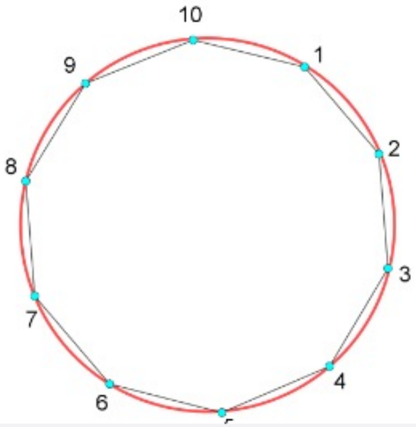
Here is an early pot from the Toya family of Jemez. Try making stars on this decagon (10-gon).











SOME MORE ADVANCED IDEAS

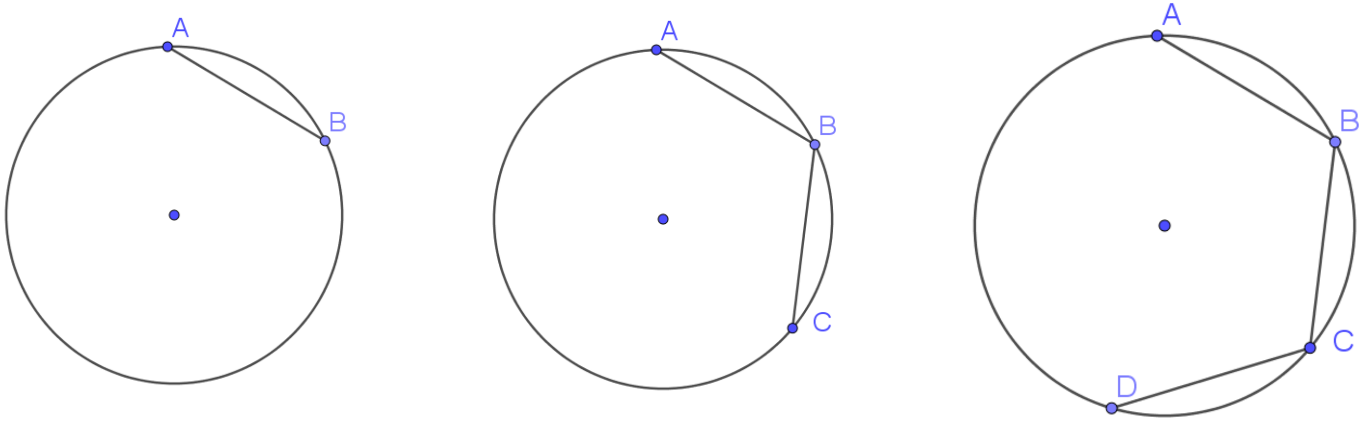
Take a circle. Fix a point A on the circle. We can start drawing polygons at A.

Now choose a (random) point B for the second vertex of the polygon. Points A and B determine a polygon in the following way:

Find a third vertex C so that arc AB = arc BC. Find a fourth vertex D so that arc BC = arc CD.

…and so on, until we get back to point A.

The polygon may be a star polygon, or a simple regular polygon



SOME QUESTIONS:

1) How can we figure out, from the positions of points A and ab, how many sides the polygon will have?

OR…

SOME MORE ADVANCED IDEAS

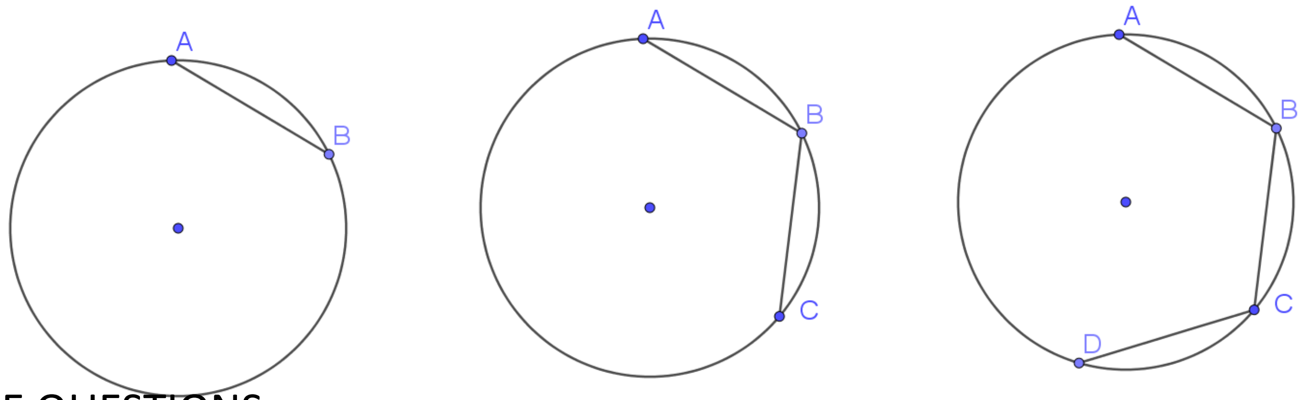
Take a circle. Fix a point A on the circle. We can start drawing polygons at A.

Now choose a (random) point B for the second vertex of the polygon. Points A and B determine a polygon in the following way:

Find a third vertex C so that arc AB = arc BC. Find a fourth vertex D so that arc BC = arc CD.

…and so on, until we get back to point A.

The polygon may be a star polygon, or a simple regular polygon



SOME QUESTIONS:

1) How can we figure out, from the positions of points A and ab, how many sides the polygon will have?

OR…

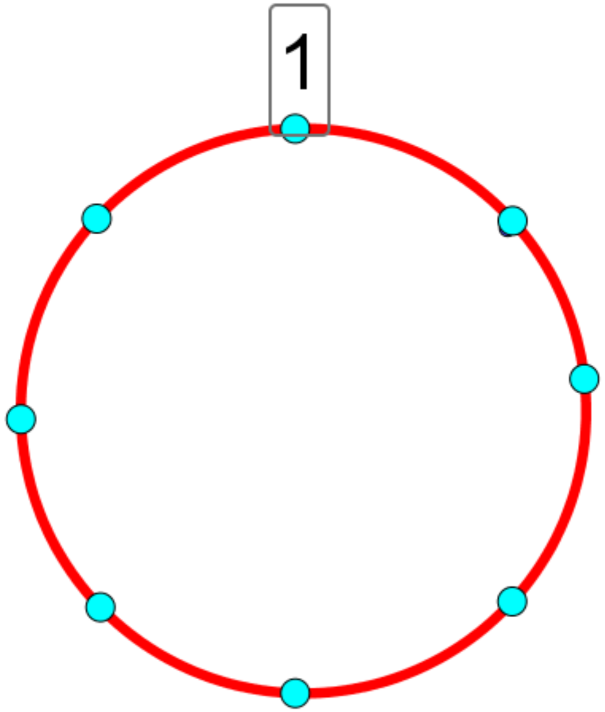
2) Or even if we will EVER come back to point A? Maybe we will go round and round, and never hit point A a second time.

That is, the ‘winding number’ of our polygon will be infinite.

Think of things this way.

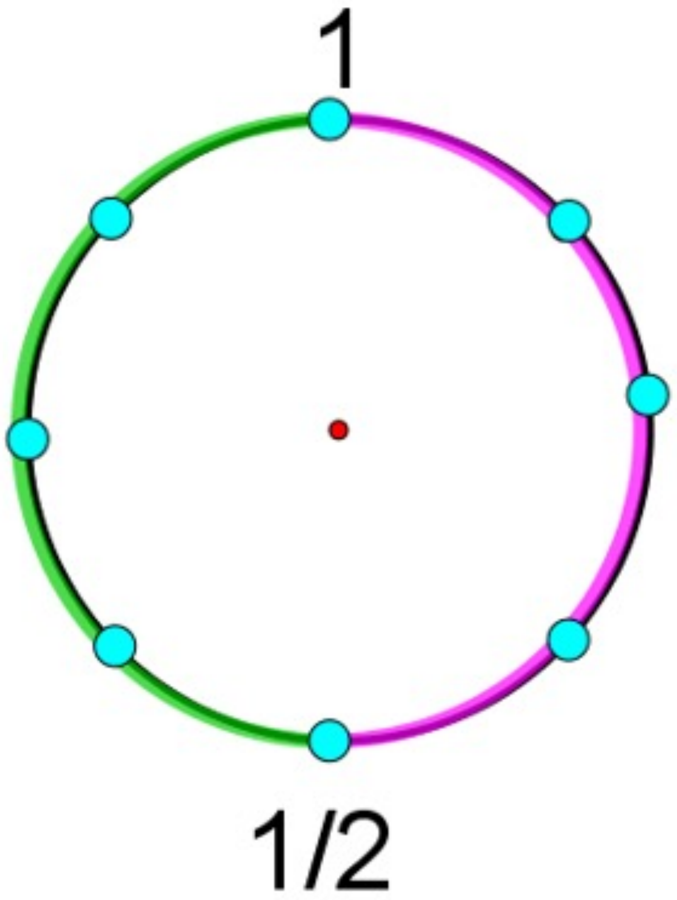
We can think of a star polygon as ‘generated’ by a single arc of a circle

For example, take the circle with 8 points on it that we started with.



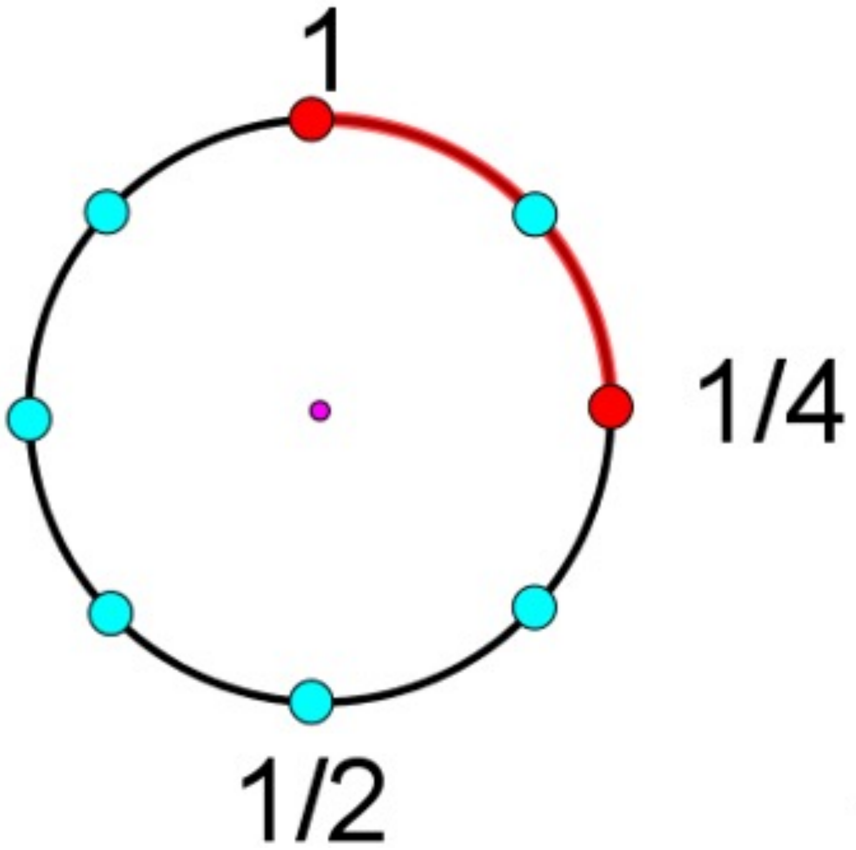
If we start at the top and go around the circle, that’ a full rotation. So this time, instead of labeling the point on top 0, we will label it 1 (for 1 full rotation).

Then the diametrically opposite point is 1/2, for 1/2 a rotation. If we lay it off along the circle a second time, we get back to 1 (rotation):



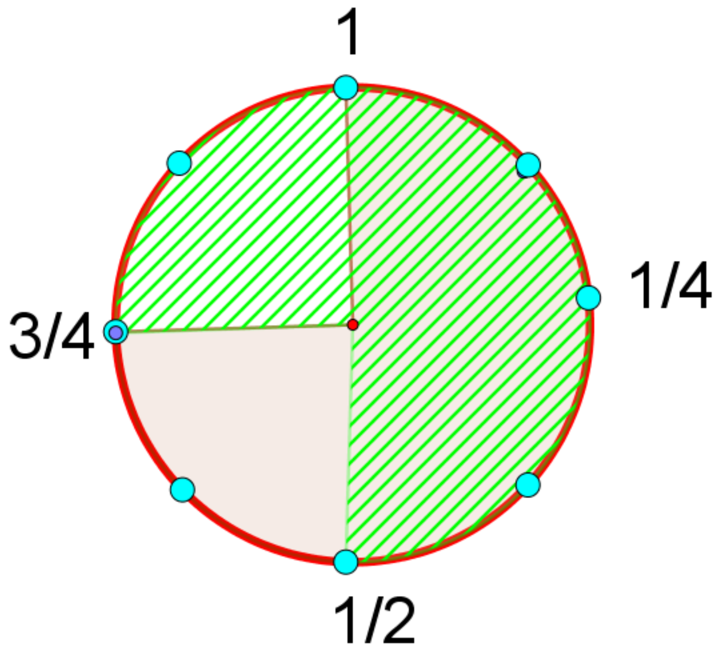
And the point in the middle of those two is ¼, for ¼ of a rotation.

That is, if we lay it off four times around the circle, we get back to 1. We have made a full rotation.



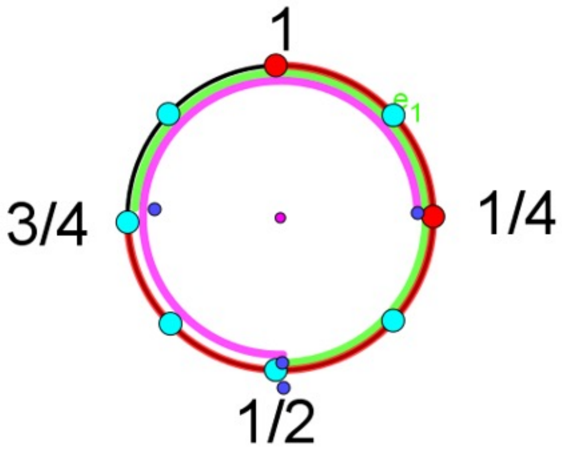
What about the other point in the middle of 1/2 and 1? That point is ¾ of the way around the circle. How many times must we lay it off around the circle to get back to 1 rotation?

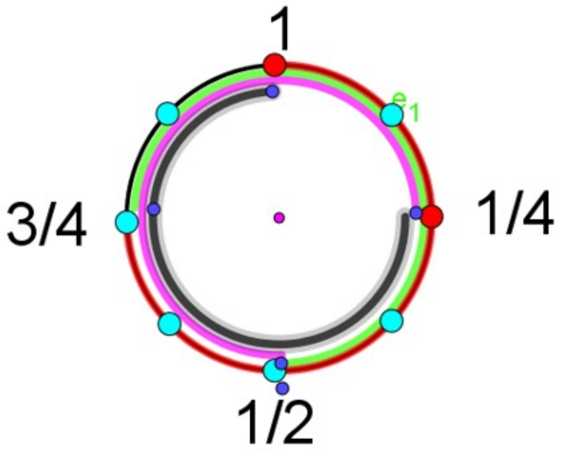
Well, a second one already goes past the 1, to the point we've labeled 1/2 . But let's go on accumulating ¾ rotations.



Three of them is 3(3/4) = 9/4 = 1 + ¼ rotations.

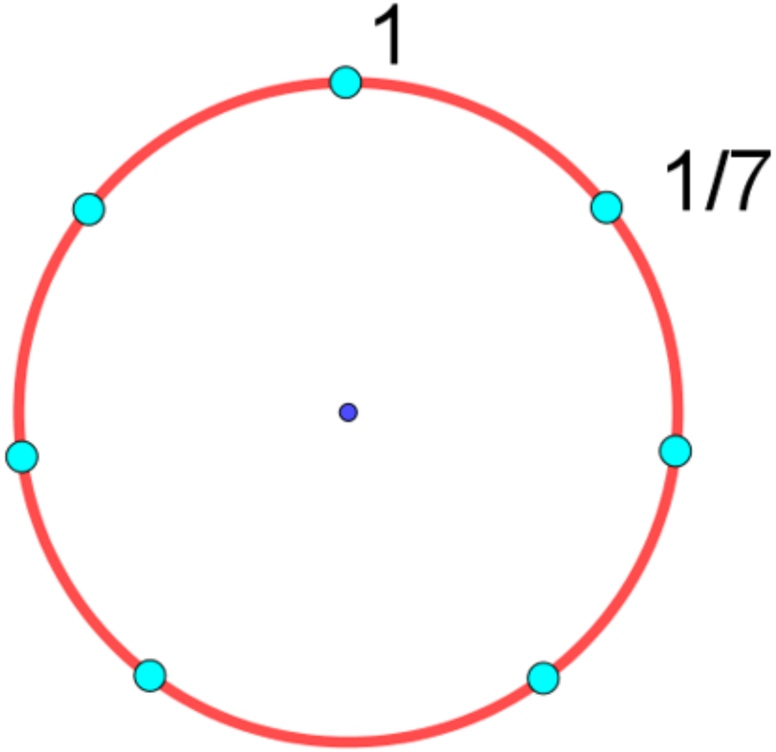
And four of them makes 2 rotations, and we are back to the point labeled 1. We haven't made just one rotation, but we have made a whole number of rotations.



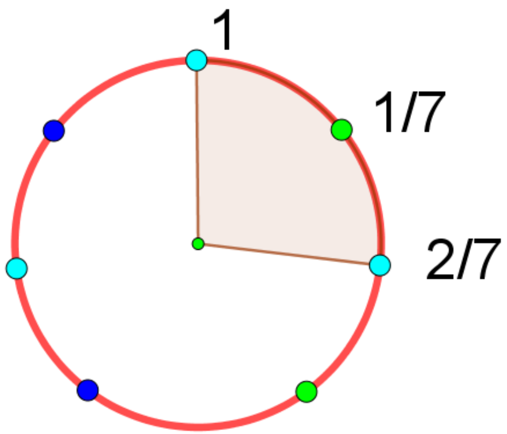


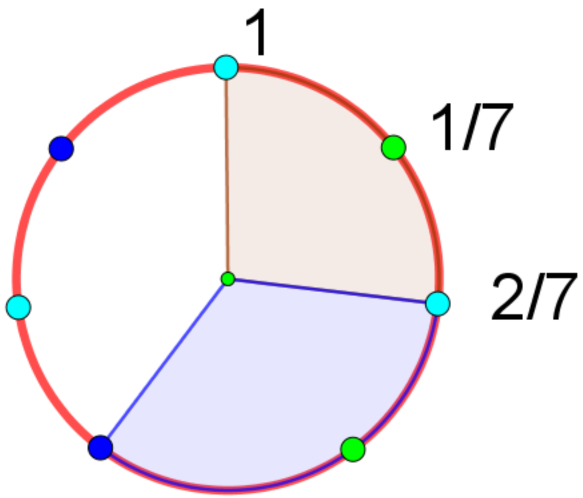
Another example: the Zuni pot led us to a circle with 7 points marked on it.

We can call the first point 1/7 of a rotation. Seven of them make a full rotation.



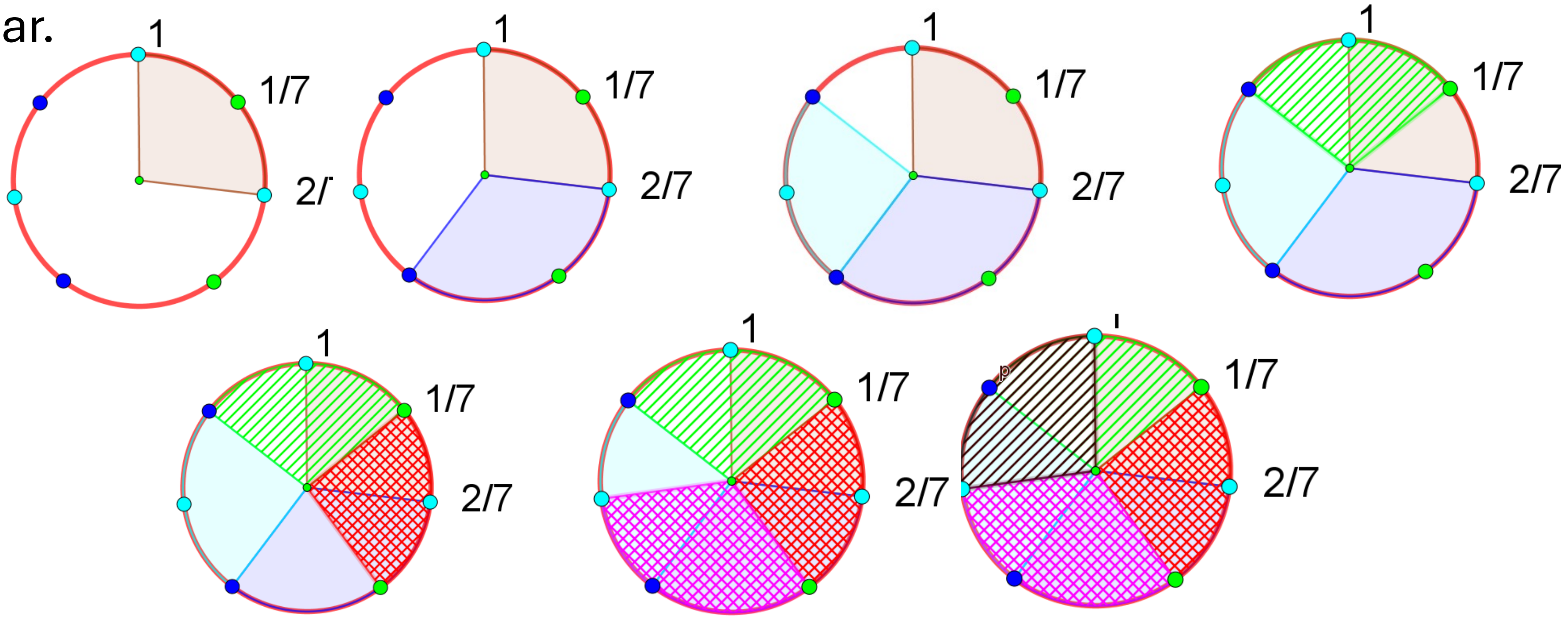
It seems reasonable to call the second point 2/7 of a rotation. Do seven of them make a full rotation?



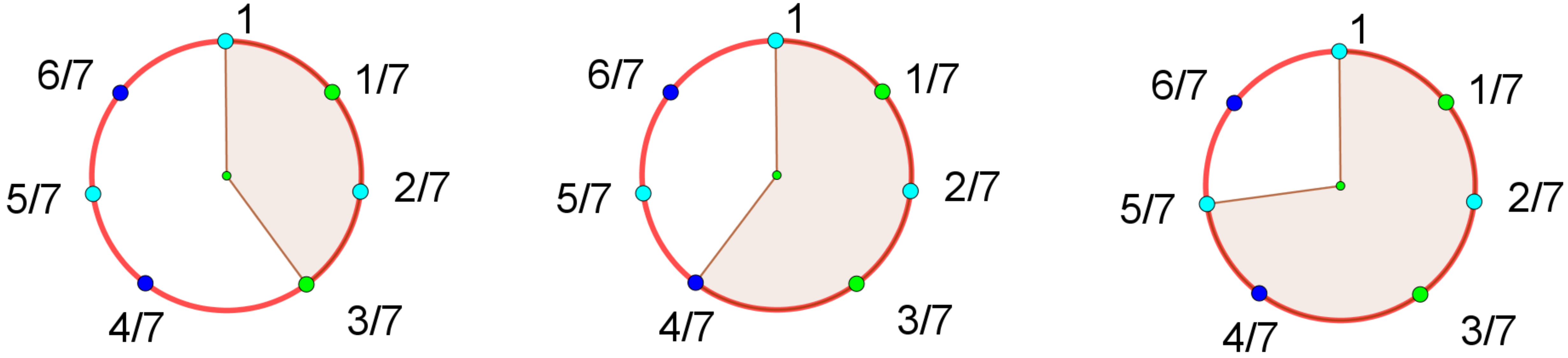


Well, seven 2/7 rotations get us back to the point labeled 1. So a 2/7 rotation doesn't give us one rotation. But it gives us a whole number of rotations.

In fact when we get back to '1' we have made two full rotations: the numerator of the fraction—and the winding number of the corresponding star.



What do you think happens with 3/7? With 4/7? With 5/7?



What do you think happens with 3/7? With 4/7? With 5/7?

With 3/7, we lay it off seven times, and we get 3 rotations: 7 (3/7) = 3.

With 4/7, we lay it off seven times and we get 4 rotations: 7 (4/7) = 4.

With 5/7, we lay it off seven times and we get 5 rotations: 7 (5/7) = 5.

What do you think happens with 3/7? With 4/7? With 5/7?

With 3/7, we lay it off seven times, and we get 3 rotations: $7\,(3/7) = 3$.

With 4/7, we lay it off seven times and we get 4 rotations: $7\,(4/7) = 4$.

With 5/7, we lay it off seven times and we get 5 rotations: $7\,(5/7) = 5$.

If we are making a star with M/7 of a rotation then:
The denominator of the fraction is the number of rotations before we 'land on' a full rotation.
The numerator of the fraction is the winding number of the star.

Is this true for other denominators? What about in general, for a fraction M/N of a rotation?

But what if you have 1/π of a rotation?

But what if you have 1/π of a rotation?

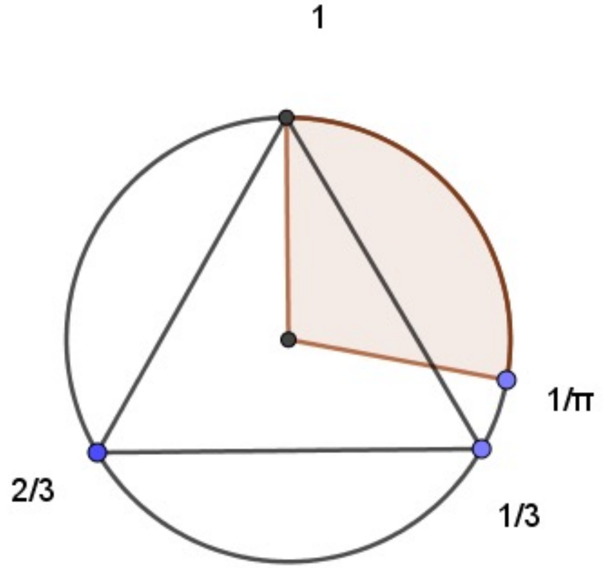
Is there such a number? Sure there is. It's approximately $\frac{1}{3.14159}$.

Is it bigger than 1/3? Or less than 1/3?

.

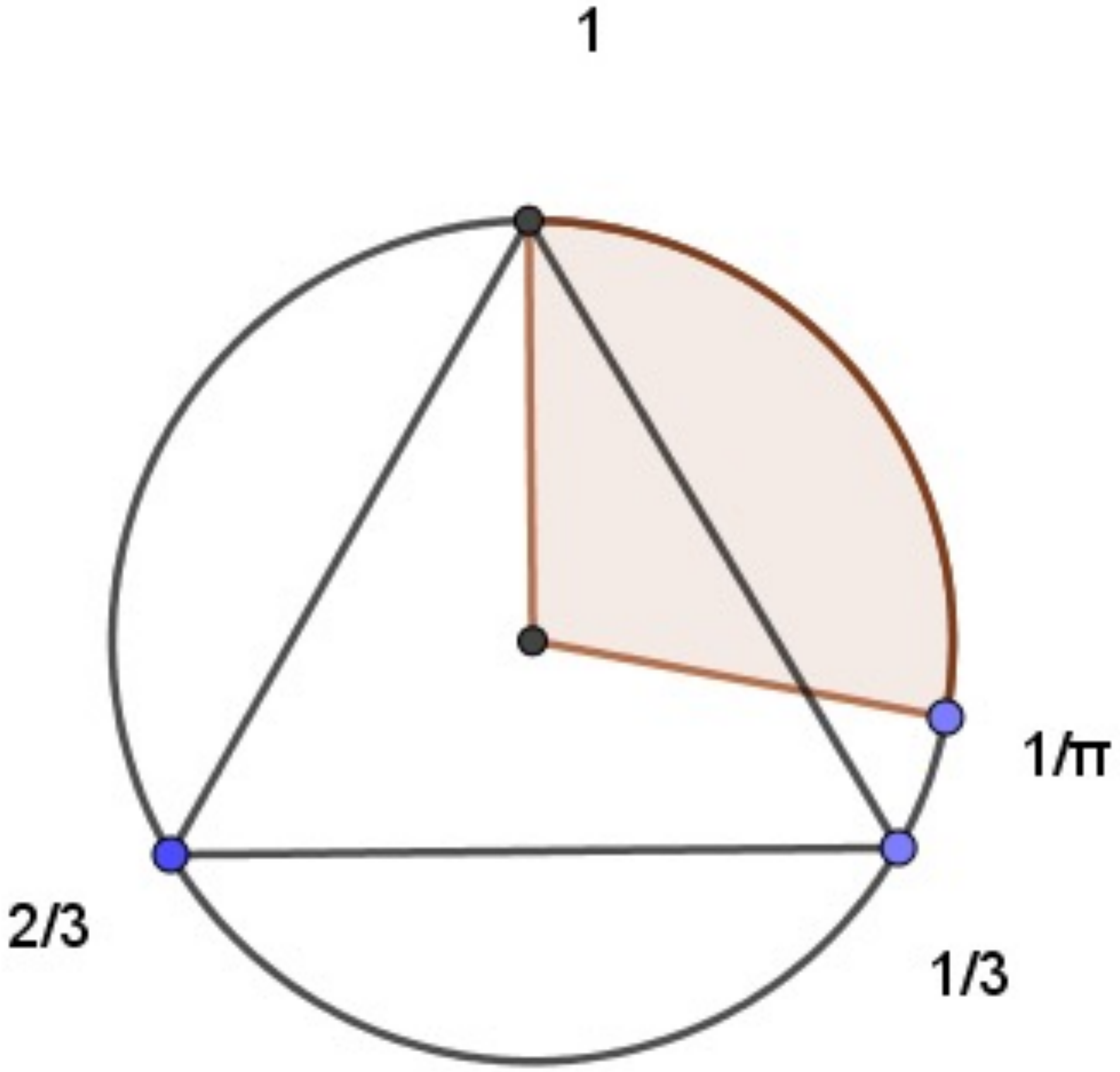
But what if you have 1/π of a rotation? Is there such a number? Sure there is. It's approximately $\frac{1}{3.14159}$.

Is it bigger than 1/3? Or less than 1/3?



It's less than 1/3: the bigger the denominator, the smaller the fraction (if the numerators are the same, as they are here).

So we can have an arc which is 1/π of a rotation. How many times must we lay it off to get back to 1?



So we have an arc which is $1/\pi$ of a rotation. How many times must we lay it off to get back to 1?

Well, suppose we lay it off M times, and as a result we get N rotations. Of course, M and N are whole numbers.

So we have an arc which is $1/\pi$ of a rotation. How many times must we lay it off to get back to 1?

Well, suppose we lay it off M times, and as a result we get N rotations. Of course, M and N are whole numbers.

Then $M(1/\pi) = N$, and $1/\pi = N/M$, So π would equal M/N.

That cannot happen! The number π is irrational!

Then M(1/π) = N, and 1/π = N/M, So π = M/N. That cannot happen! The number π is irrational!

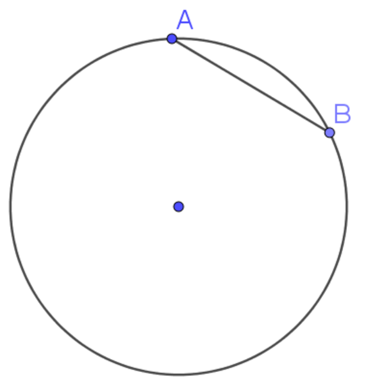
Recall that a rational number is one which can be represented as a fraction, with integer numerator and denominator:

$$\frac{2}{3} \qquad \frac{-3}{7} \qquad -\frac{4}{5} \qquad \frac{12345}{23456}$$

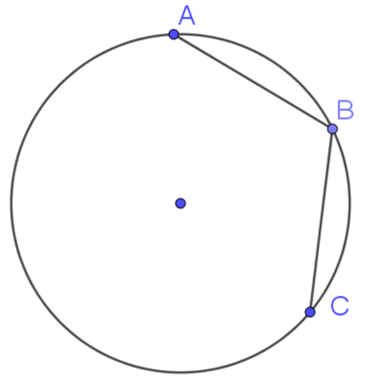
But some numbers, like π, cannot be represented as such a fraction.

IN GENERAL

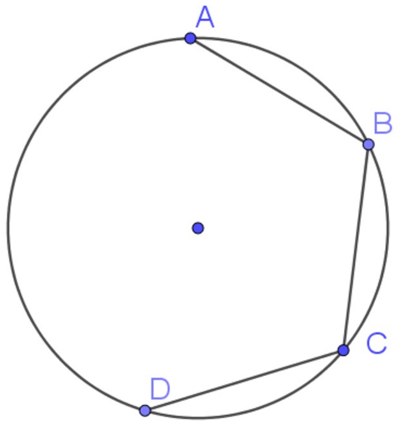
Take a circle. Fix a point A on the circle. We can start drawing polygons at A.



Now choose a (random) point B for the second vertex of the polygon. Points A and B determine a polygon in the following way:



Find a third vertex C so that arc AB = arc BC.
Find a fourth vertex D so that arc BC = arc CD.



…and so on, until we get back to point A.

The polygon may be a star polygon, or a simple regular polygon…

SOME MORE ADVANCED IDEAS

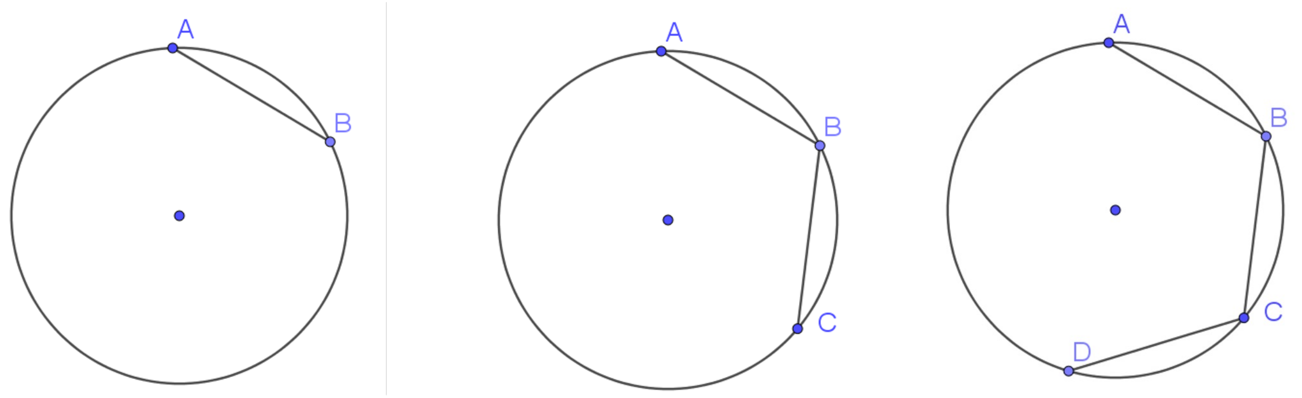
Take a circle. Fix a point A on the circle. We can start drawing polygons at A.

Now choose a (random) point B for the second vertex of the polygon. Points A and B determine a polygon in the following way:

Find a third vertex C so that arc AB = arc BC.
Find a fourth vertex D so that arc BC = arc CD.

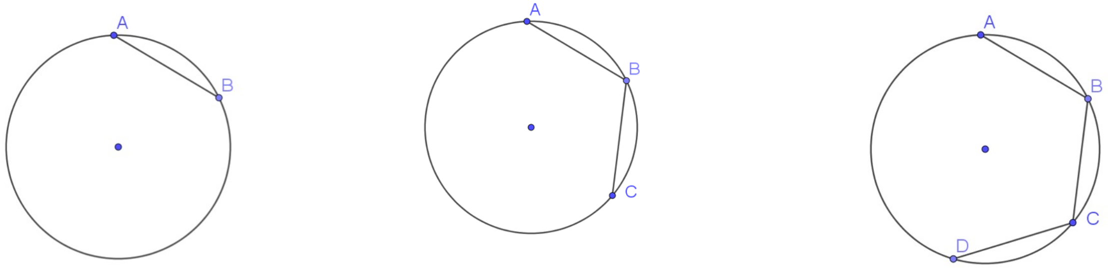
…and so on, until we get back to point A.

The polygon may be a star polygon, or a simple regular polygon



Or, if arc AB is an irrational part of the full circumference, it may never close!

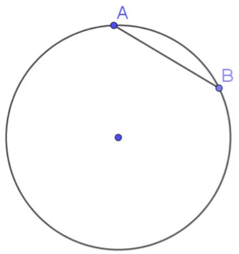
The ‘winding number’ may be infinite!

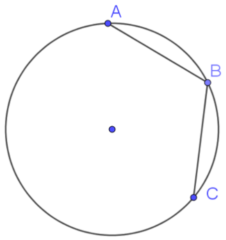


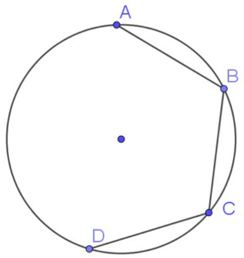
Suppose we pick points A and B, and it happens that, when we draw the polygon it generates, the polygon does not ‘close’: we never get back to point A.

What will the diagram look like? Which points on the circle will end up as vertices of the polygon? Which will not?

This question is answered by some interesting theorems in *group theory.*





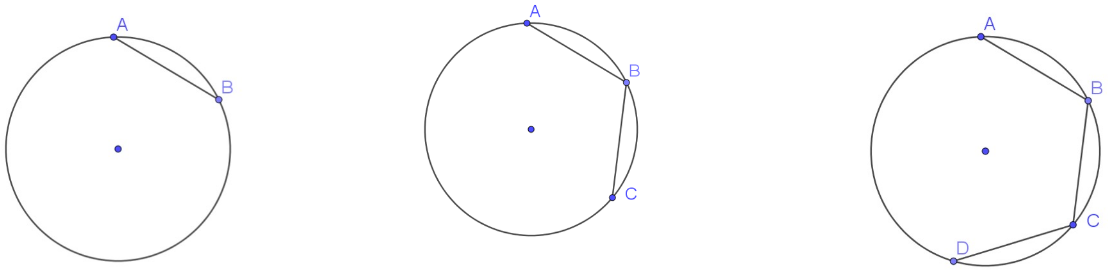


3) Suppose we pick points A and B, and it happens that, when we draw the polygon it generates, the polygon does not ‘close’: we never get back to point A.

What will the diagram look like? Which points on the circle will end up as vertices of the polygon? Which will not?

This question is answered by some interesting theorems in *group theory.*

We don’t have time to go into these theorems. But meanwhile, you can think how the following question ends up being related to our exploration of star polygons. (The question is on the next slide.)

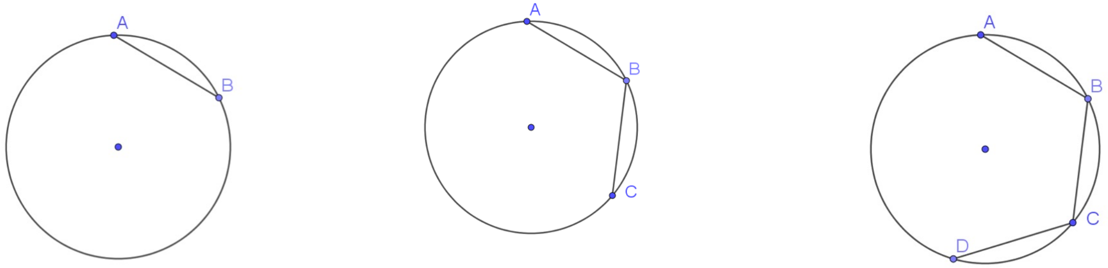


3) Suppose we pick points A and B, and it happens that, when we draw the polygon it generates, the polygon does not ‘close’: we never get back to point A.

What will the diagram look like? Which points on the circle will end up as vertices of the polygon? Which will not?

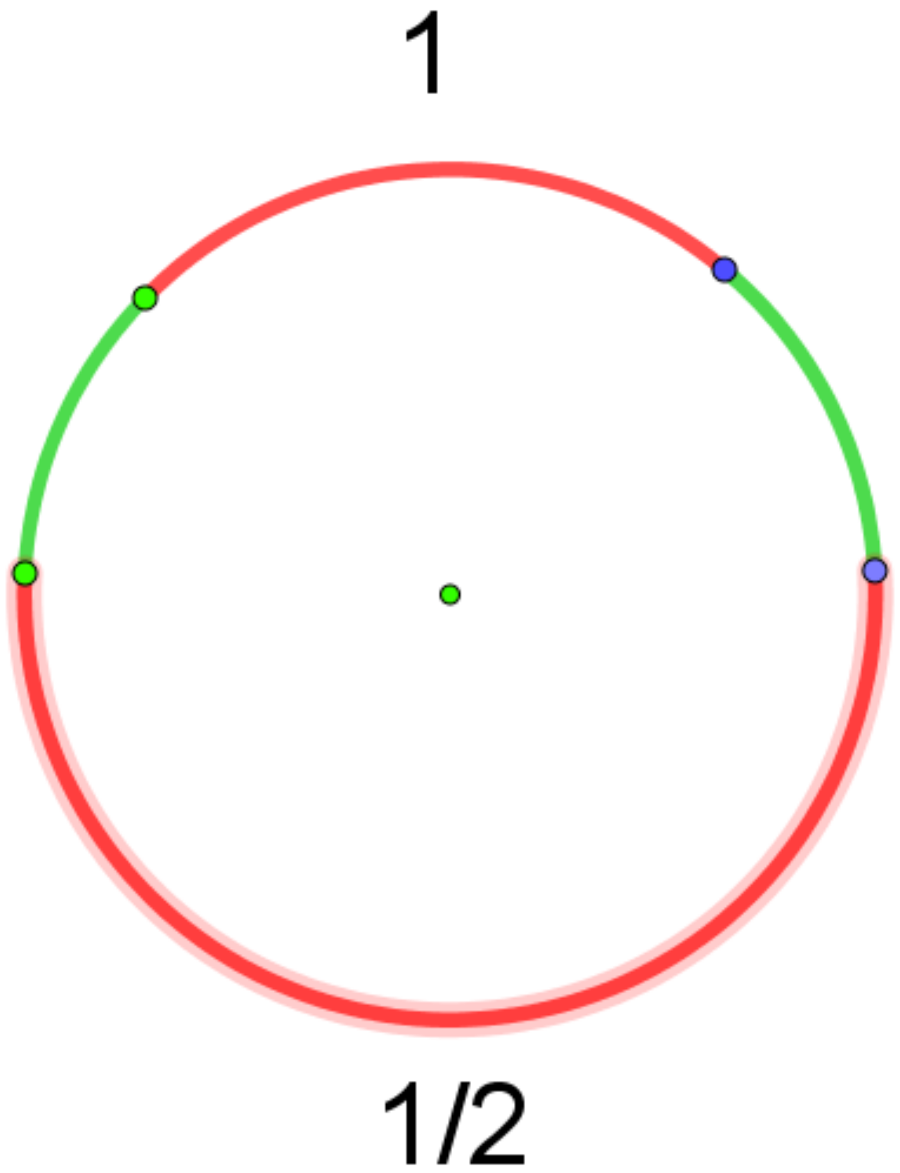
Will the points be ‘evenly distributed’ around the circle?

Or will there be patches of the circle where none of these rotated points will land?



3) Suppose we pick points A and B, and it happens that, when we draw the polygon it generates, the polygon does not ‘close’: we never get back to point A.

What will the diagram look like? Which points on the circle will end up as vertices of the polygon? Which will not?



Will the points be ‘evenly distributed’ around the circle?

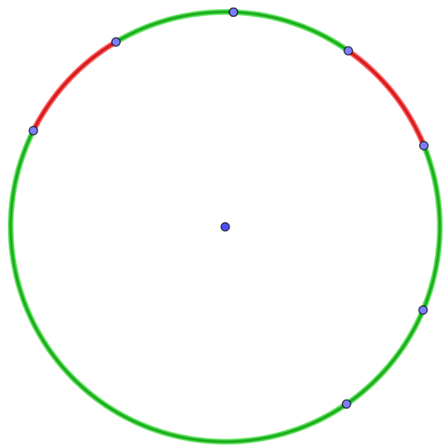
Or will there be patches of the circle where none of these rotated points will land?

Perhaps the situation is like the picture at right: there are points inside the green areas, but none inside the red areas.

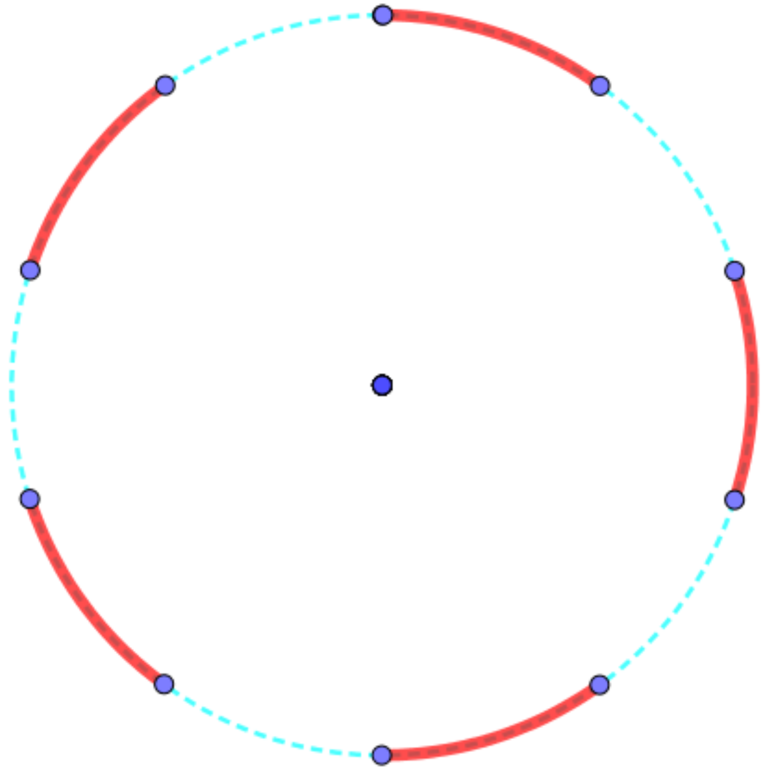
Will the points be ‘evenly distributed’ around the circle?

Or will there be patches of the circle where none of these rotated points will land?

Perhaps the situation is like the picture at right: there are points inside the green areas, but none inside the red areas.



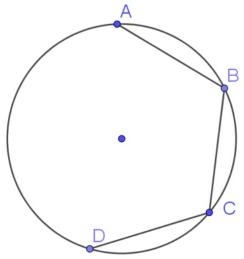
Or the picture might look even worse:



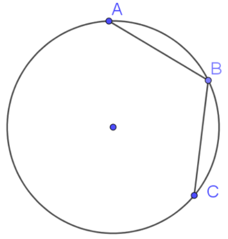
It turns out that either the points will form a star polygon, or the process will go on forever. If the latter, the points will be distributed evenly around the circle.

This thought is made very precise by theorems in higher mathematics.

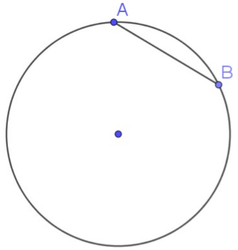
And here is an unexpectedly related question:



Look at the sequence of powers of 2:

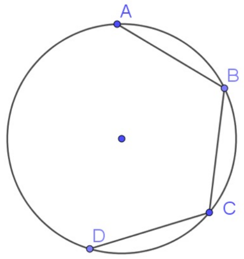


1
2
4
8

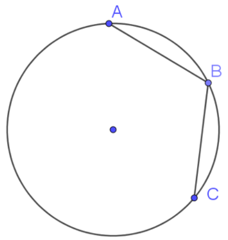


16
32
64
128
256
512
1024
2048
….

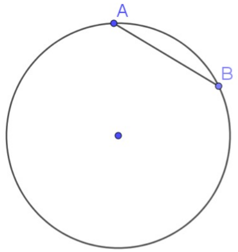
And here is an unexpectedly related question:



Look at the sequence of powers of 2:



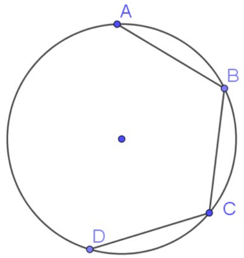
1
2
4
8
16
32
64
128
256
512
1024
2048
….



Warning: Logarithms coming up.

But don’t worry, you’ll be able to understand the question. And when you learn about logarithms, you’ll be able to understand the answer.

And here is an unexpectedly related question:

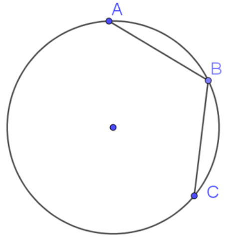


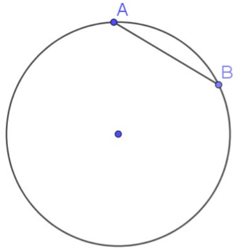
Look at the sequence of powers of 2:

1
2
4
8
16
32
64
128
256
512
1024
2048
….

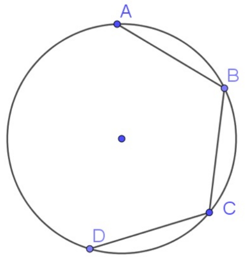
If you look at the rightmost digits, you will see that they repeat in a cycle of four (after the initial ‘1’):

2, 4, 8, 6, 2, 4, 8, 6,….





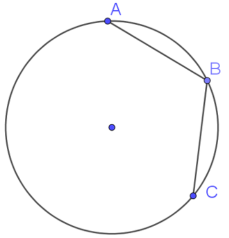
And here is an unexpectedly related question:



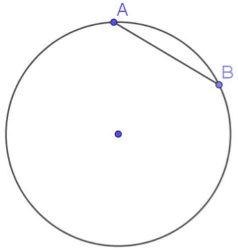
Look at the sequence of powers of 2:

1
2
4
8
16
32
64
128
256
512
1024
2048
....

If you look at the rightmost digits, you will see that they repeat in a cycle of four (after the initial ‘1’):



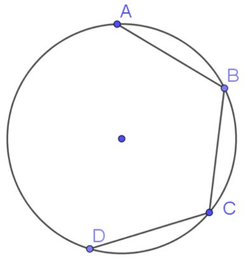
2, 4, 8, 6, 2, 4, 8, 6,....



And if you think about how the sequence is generated, by ‘doubling’, you can convince yourself that this cycle MUST repeat. Each’period’ of four numbers is determined by the previous.

This process is made formal by the beautiful “Principle of Mathematical Induction,” which you will explore in more advance courses.

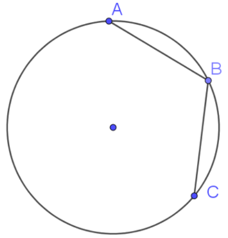
And here is an unexpectedly related question:



Look at the sequence of powers of 2:

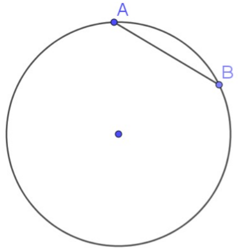
1
2
4
8
16
32
64
128
256
512
1024
2048
….

If you look at the rightmost digits, you will see that they repeat in a cycle of four (after the initial '1'):

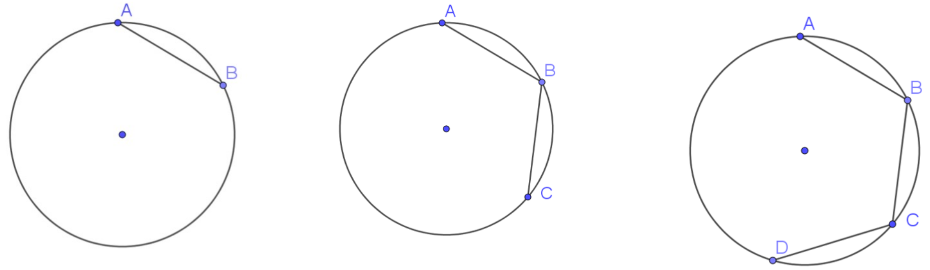


2, 4, 8, 6, 2, 4, 8, 6,….

(That's not the "unexpectedly related question."



Here is the unexpectedly related question:



Look at powers of 2:

1
2
4
8
16
32
64
128
256
512
1024
2048 ….

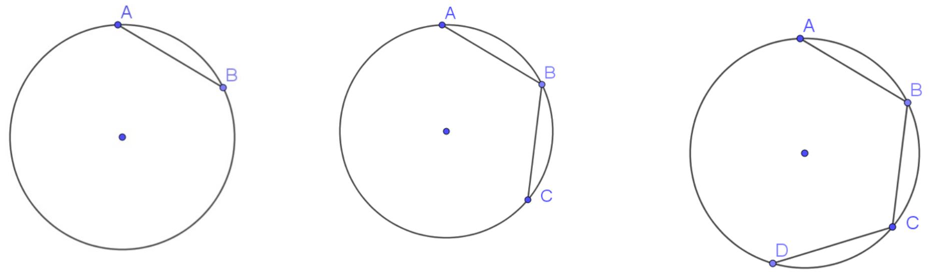
This time look at the sequence of their first (leftmost) digits:

1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2. . . .

a) Does this sequence repeat?
b) Will it ever contain the digit 7?

For example, does there exist a power of 2 whose leftmost digit is 7?

Here is an unexpectedly related question:



Look at powers of 2:

01
02
04
08
16
32
64
128
256
512
1024
2048 ….

…And look at the sequence of their first (leftmost) digits:

1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2. . . .

a) Does this sequence repeat?
b) Does it ever contain the digit 7?

That is, does there exist a power of 2 whose leftmost digit is 7?

Now look at the sequence of the first two digits:
01, 02, 04, 08, 16, 32, 64, 12, 25, 51, 10, 20. . .

c) Does this sequence contain the number 77?

That is: Does there exist a power of 2 whose leftmost two digits are 77?

Can a power of 2 begin with the digit 7?

Here is the relationship to the pottery designs:

If 2^n begins with the digit 7, then we must have

$7x10^k < 2^n < 8x10^k$, for some integer k.

Taking logarithms (base 10), this means that

$k + \log 7 < n \log 2 < k + \log 8$, for integers k and n, or

$\text{Log } 7 < n \log 2 – k < \log 8$.

$k + \log 7 < n \log 2 < k + \log 8$, for integers k and n, or

$\text{Log } 7 < n \log 2 – k < \log 8$.

Now k is any integer, and we can actually eliminate it by taking ‘fractional parts’ of each number.

For positive numbers (all we care about), this just means the part of the number to the right of the decimal point.

For example, the fractional part of 3.14159 is just .14159

Taking logarithms (base 10), this means that

$k + \log 7 < n \log 2 < k + \log 8$, for integers k and n, or

$\log 7 < n \log 2 - k < \log 8$.

Now k is any integer, and we can actually eliminate it by taking ‘fractional parts’ of each number. For positive numbers (all we care about), this just means the part of the number to the right of the decimal point.

For example, the fractional part of 3.14159 is just .14159.

We write this as $\{3.14159\} = .14159$.

So we can write:

$$\{\log 7\} < \{n \log 2 - k\} < \{\log 8\}$$

Taking logarithms (base 10), this means that

$k + \log 7 < n \log 2 < k + \log 8$, for integers k and n, or

$\log 7 < n \log 2 - k < \log 8$.

So we can write:

$\{\log 7\} < \{n \log 2 - k\} < \{\log 8\}$

BUT log 7 and log 8 are both between 0 and 1, so their fractional parts are just the numbers themselves.

And subtracting an integer from a number does not change its fractional part. For example:

$\{3.14159 - 1\} = \{2.14159\} = .14159$.

So we can forget about subtracting k (!!!)

That is, the statement

“$7x10^k < 2^n < 8x10^{(k)}$, for some integers n and k”

is equivalent to the statement

“$\log 7 < \{n \log 2\} < \log 8$, for some integer n”.

This makes sense:

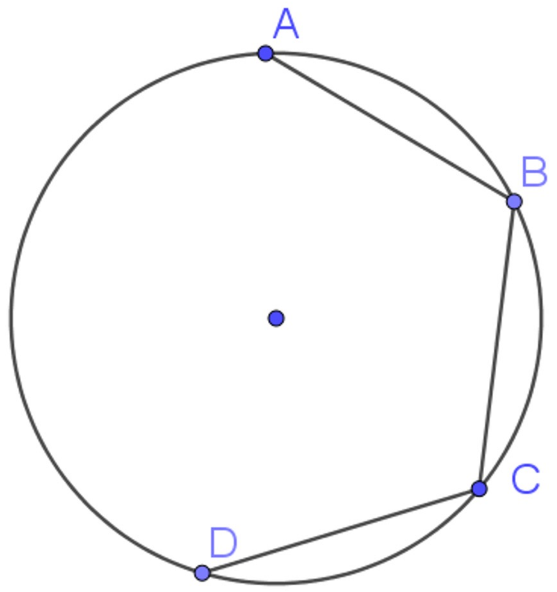
Multiplying by 2, we add log 2 to the number we start with. So we are just asking whether there’s an integer multiple of log 2 whose fractional part lies within a certain interval.

That is, the statement

$7x10^k < 2^n < 8x10^{(k)}$, for some integers n and k.

is equivalent to the statement

log 7 < n log 2 < log 8, for some integer n.



This makes sense:

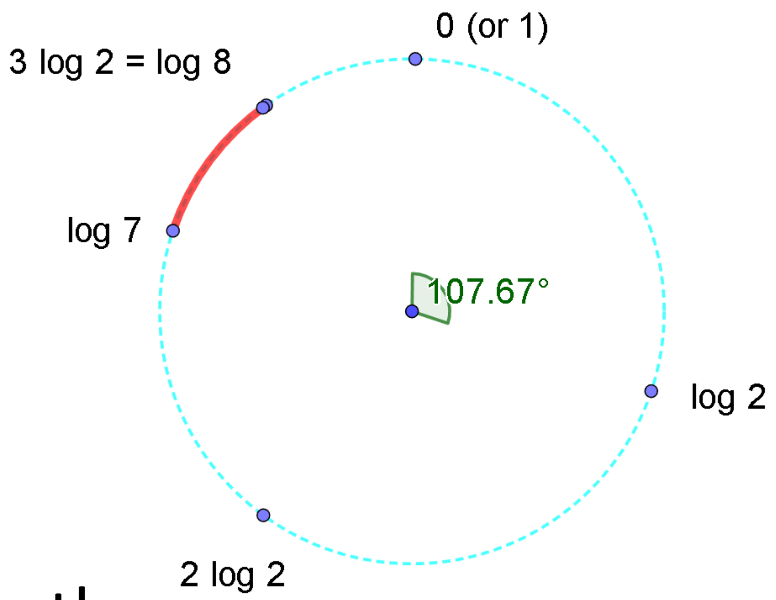
If we multiply a number by 2, we add log 2 to it is logarithm. So we are just asking whether there's an integer multiple of log 2 whose fractional part lies within a certain interval.

Now the fractional part of a number 'starts over' as soon as the fractional part grows bigger than 1. So we can think of this problem as going around a circle, forming a 'star polygon'.

We wrap the numbers between 0 and 1 around a circle. If A is the point 0 (and also the point 1), and if B is the point log 2, we are just finding points C, D, etc. which are multiples of log 2 around the circle.

Now the fractional part of a number 'starts over' as soon as the fractional part grows bigger than 1. So we can think of this problem as going around a circle, forming a 'star polygon'.

We wrap the numbers between 0 and 1 around a circle. If A is the point 0 (and also the point 1), and if B is the point log 2, we are just finding points C, D, etc. which are multiples of log 2 around the circle.



And we want to know if one of these points lands in the arc between log 7 and log 8.

That is, this is the same question that we asked about star polygons that go on forever.

The answer is given by an advanced theorem in the theory of groups, which you can look up yourself if you've followed this far!

Four boys and four girls line up in front of the class.

The boys face left and the girls face right.

Then there is a contest: who scores more, boys or girls?

Four boys and four girls line up in front of the class.

The boys face left and the girls face right.

Then there is a contest: who scores more, boys or girls?

The score for each student is the number of students he or she sees in front of him.

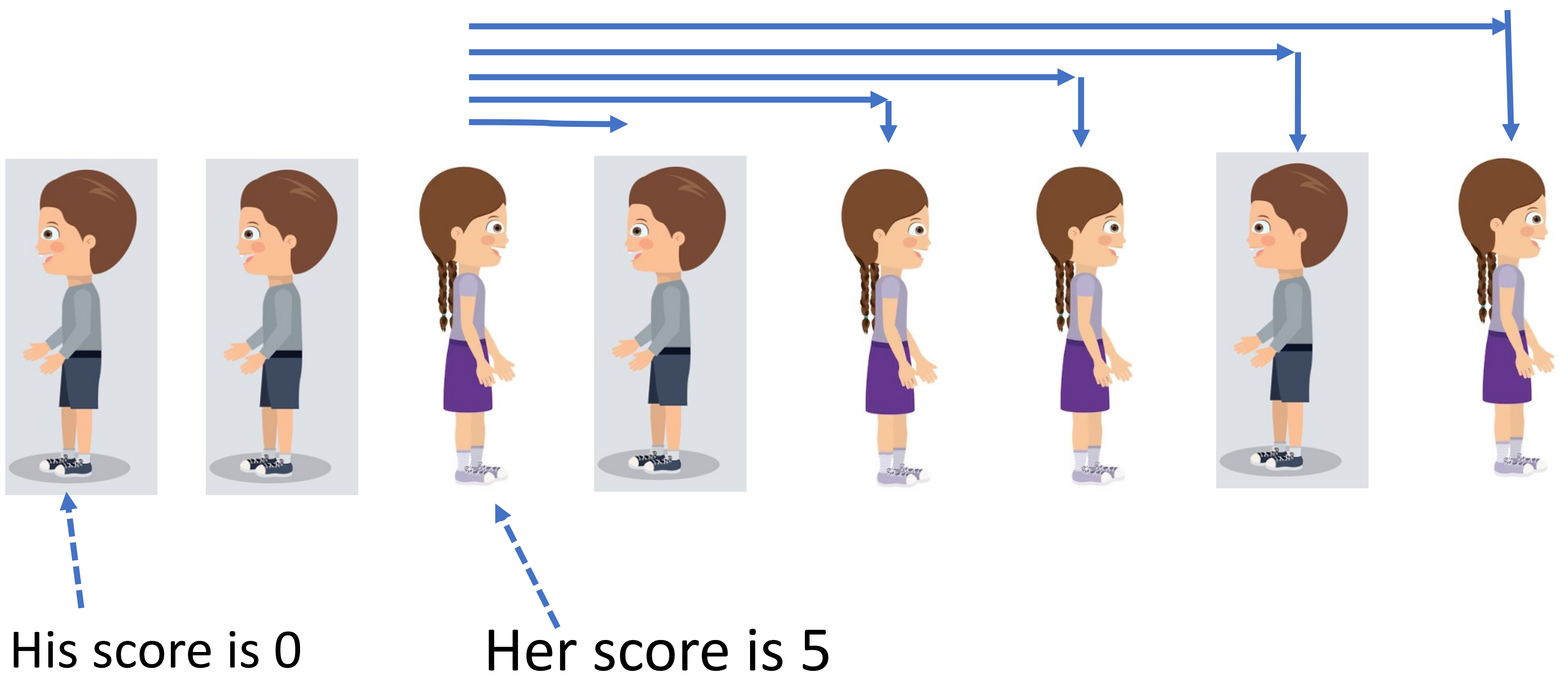
Four boys and four girls line up in front of the class.

The boys face left and the girls face right.

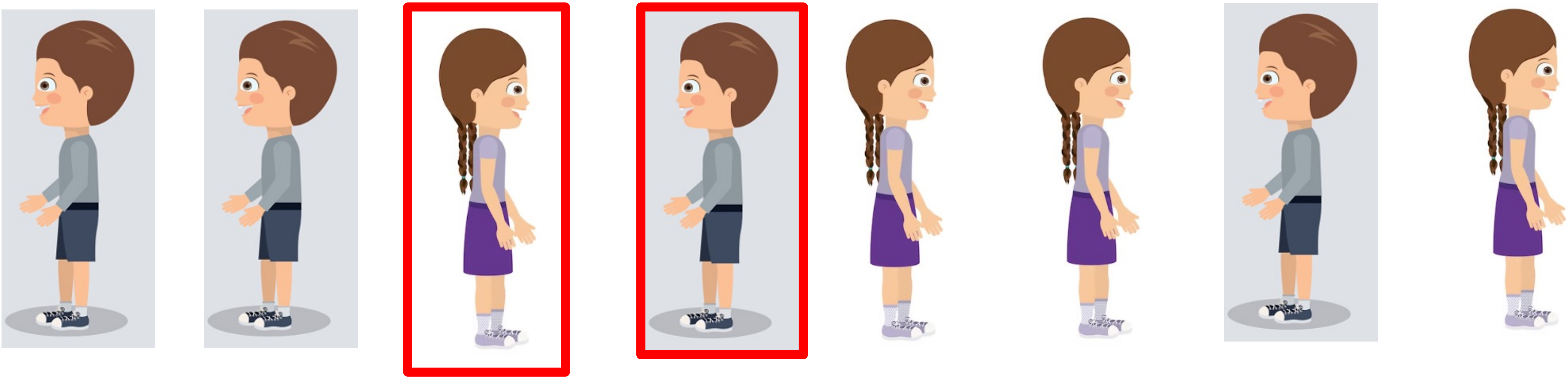
Then there is a contest: who scores more, boys or girls?

The score for each student is the number of students he or she sees in front of him.

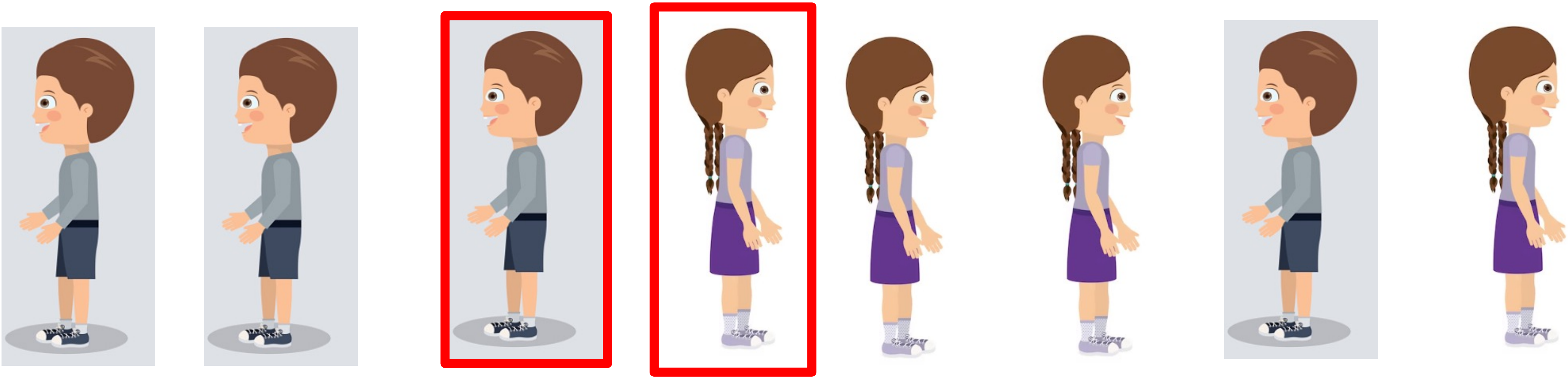
The score for the girls is the sum of each girl's score.
The score for the boys is the sum of each boy's score.

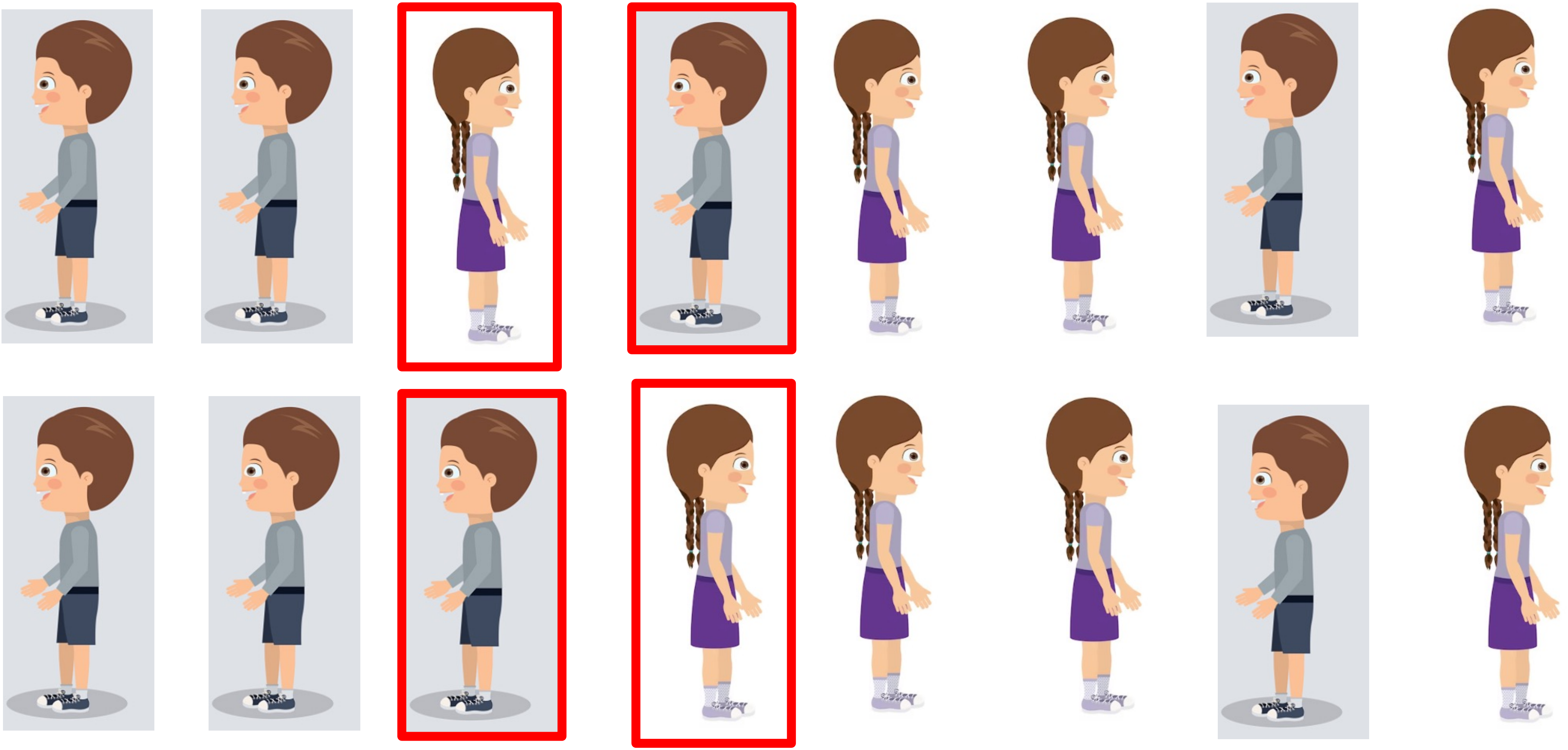

His score is 0
Her score is 5

There are several ways to prove that the score will be tied

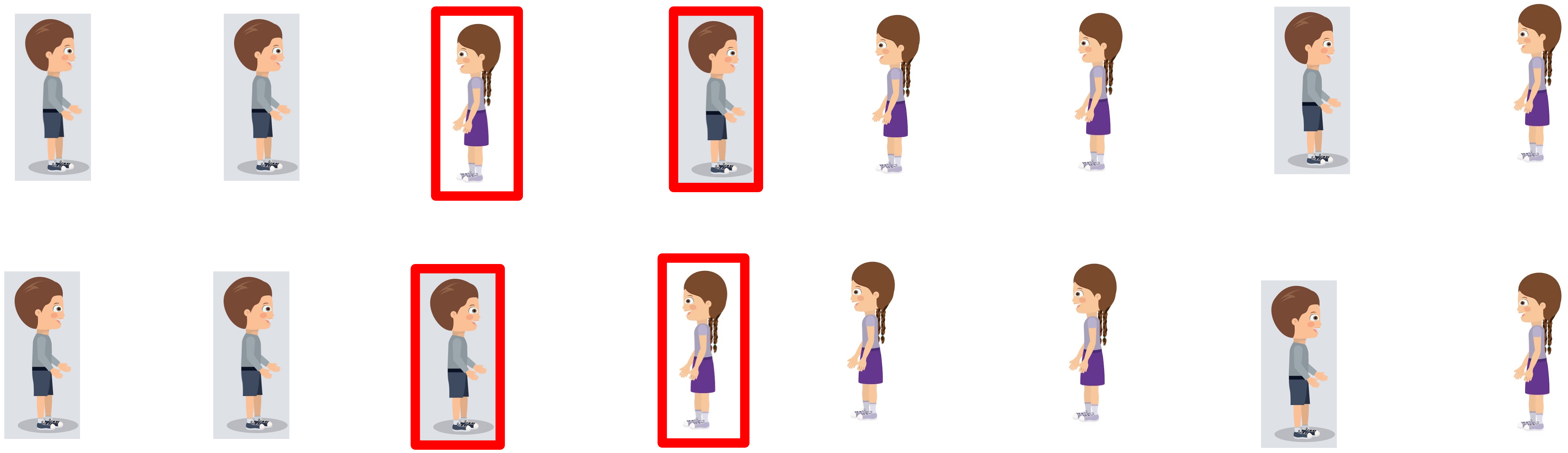


One way to show that the score will always be tied is to notice what happens when we exchange two students who are facing each other directly:





The score for each student decreases by 1.



So the difference in the boys' and girls' scores remains the same, although each score decreases by one.

If we continue this process, we get to the arrangement above, where both scores are 3+2+1 = 6.

Since the difference in scores has remained invariant, the scores must have been the same to start.

But that’s not the end!

We can ask many more questions:

>What is the largest and smallest possible tie score for 4 boys and 4 girls? What about 5 boys and 5 girls?

>Suppose there were 4 boys and 3 girls. Who would win?
There are more boys to see the girls, so maybe the boys have an advantage.
But there are more boys being looked at by girls, so maybe the girls have an advantage.

>Either way, what is the largest difference in scores that can occur?
What is the smallest difference?

Most ‘proofs’ of the original statement lead to analyses of the other statements.

‘Proof’ is ‘insight’.

It’s not just something insisted on by math teachers.

https://stock.adobe.com/search/images?k=insight+icon&asset_id=537569039