Gridlocked: Taxicab Geometry and Voronoi Diagrams

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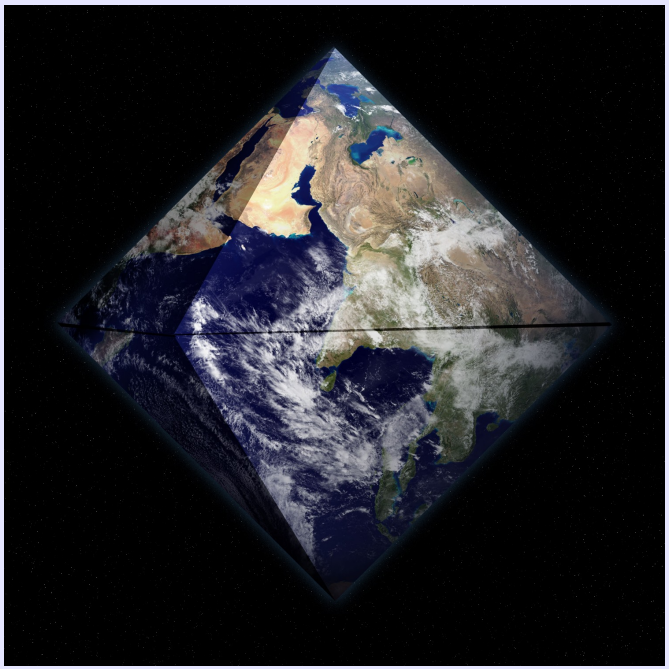
University of Puget Sound



Meet Pumpkin π

Where did Pumpkin π get his middle name?

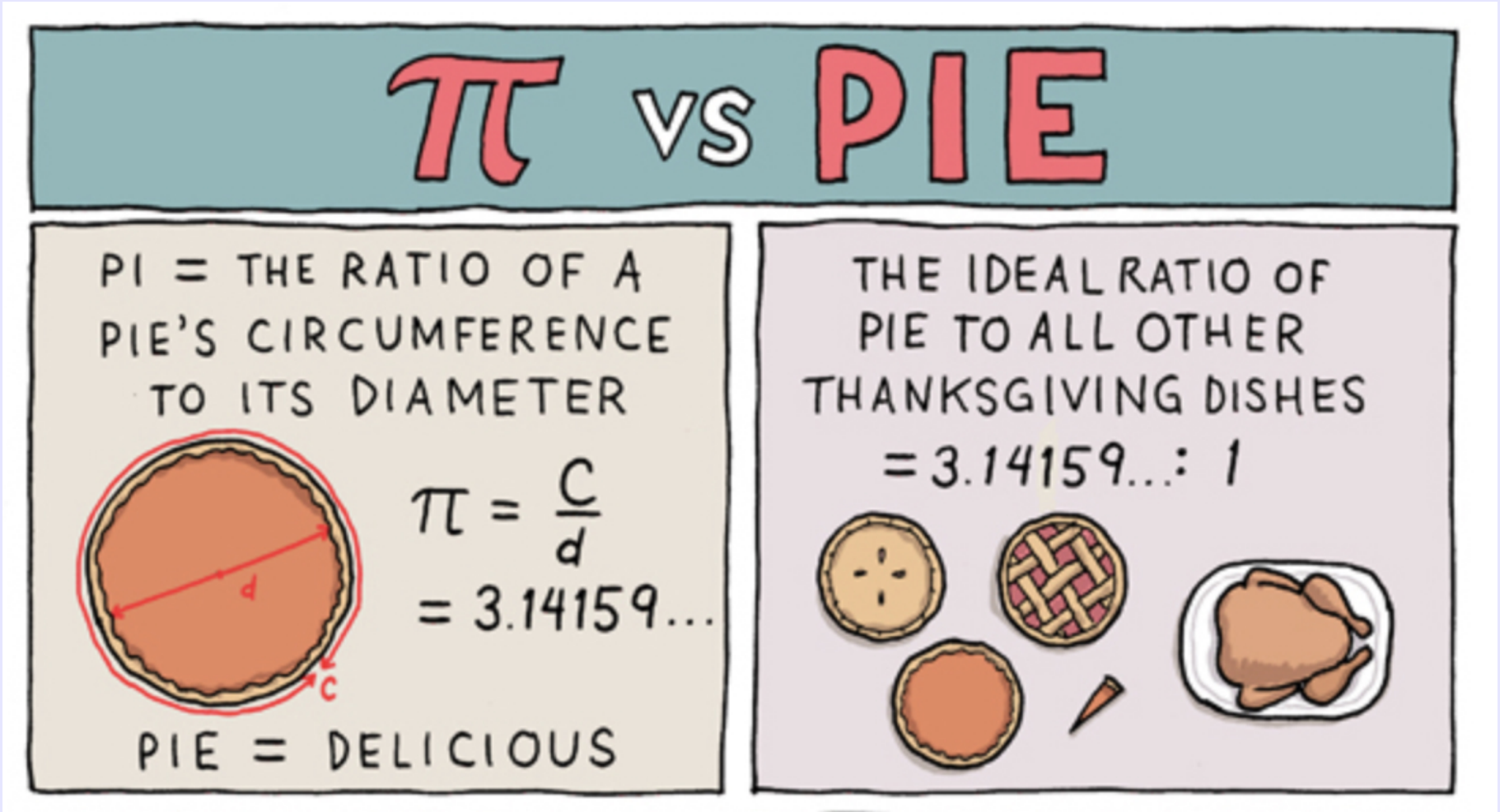
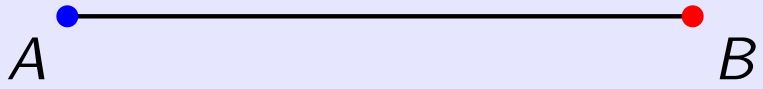


Figure: Grant Snider of Incidental Comics

Euclidland

The shortest path between two points is a straight line.



Journey to Gridland

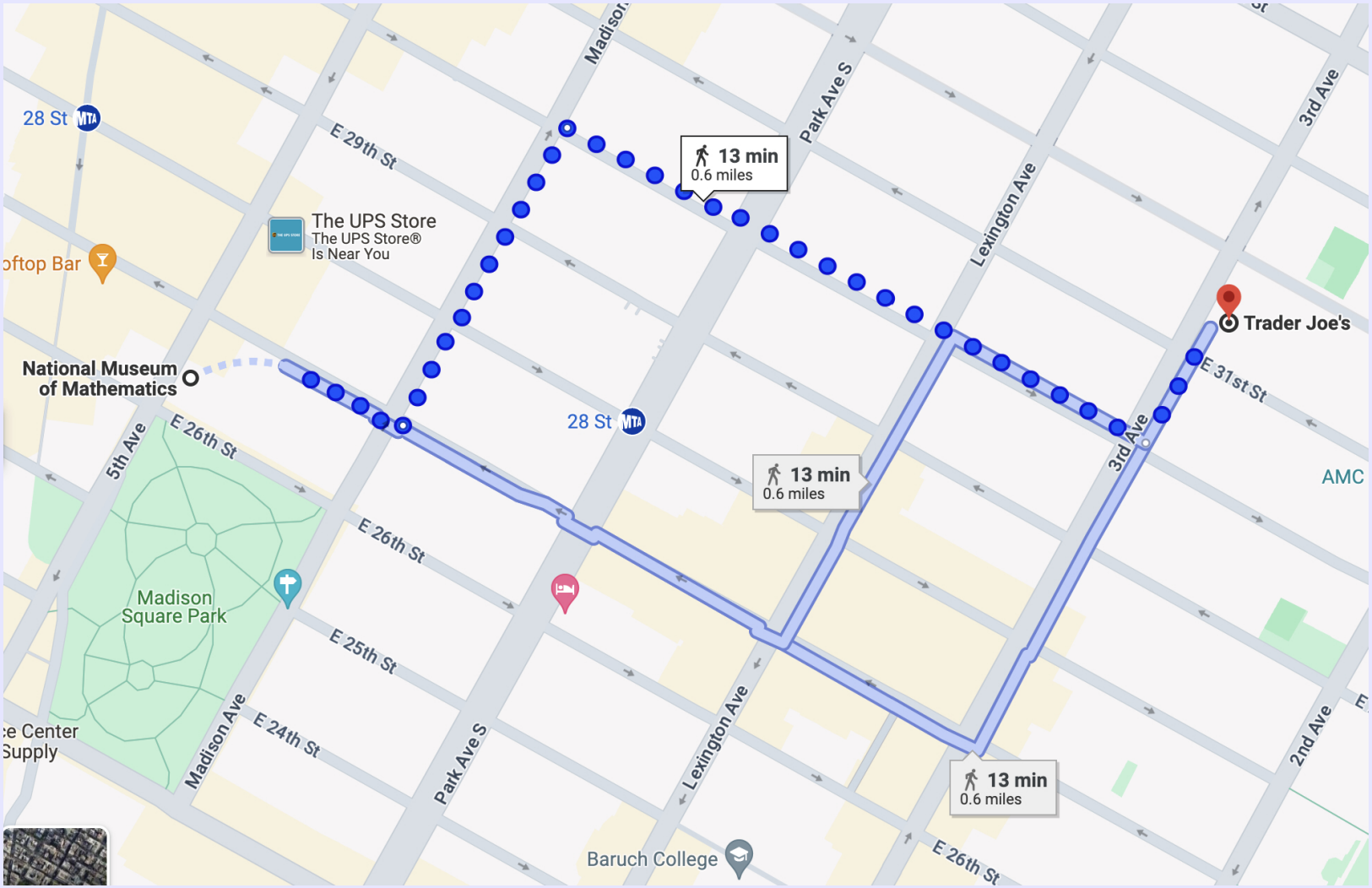


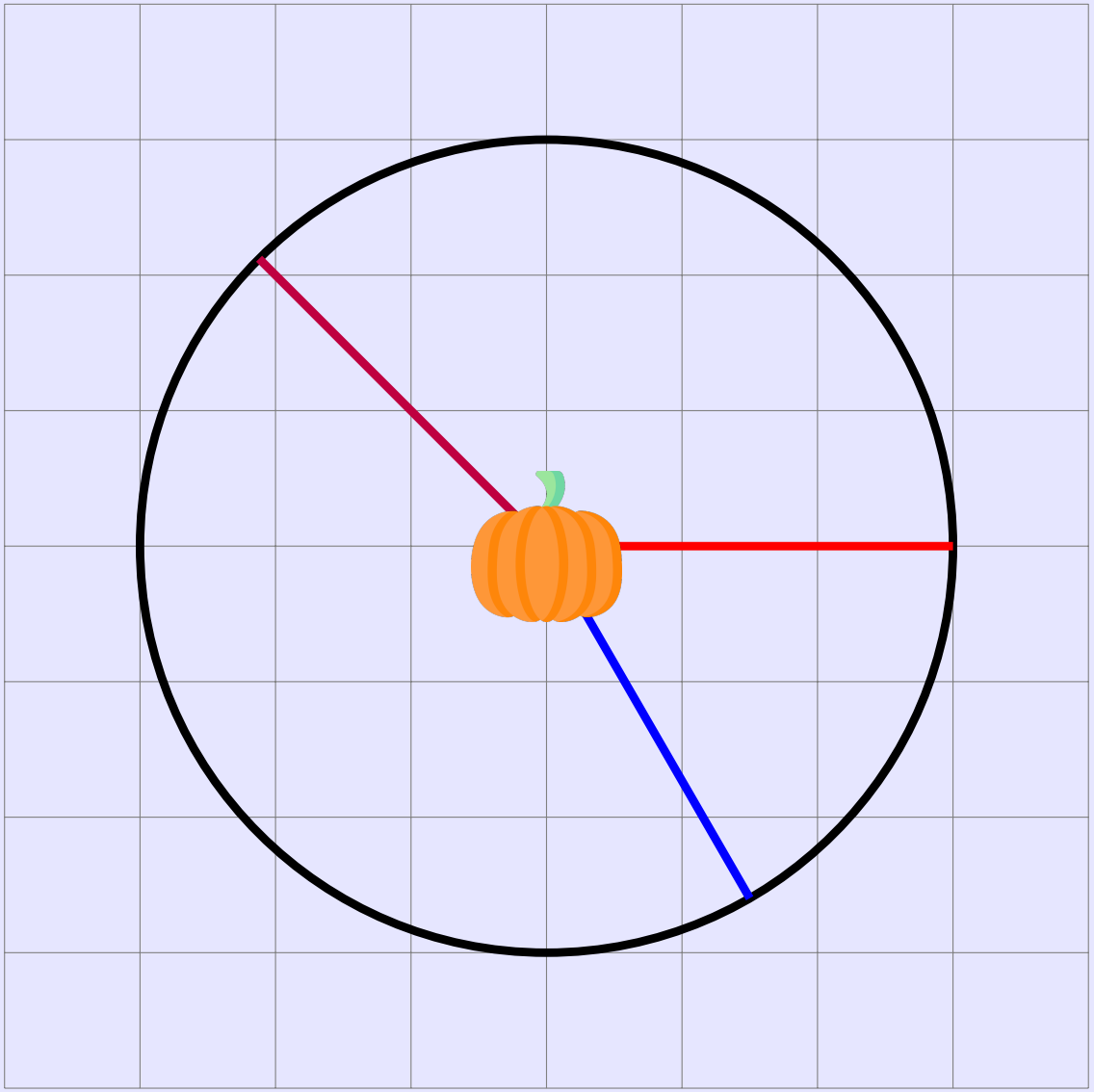
Figure: In Gridland, one cannot move along diagonals.

Leash Access Region in Euclidland

Pumpkin π is on a 3 foot leash tied to an anchor in Euclidland. He can walk in any direction. What is the shape of the region he can access?

Leash Access Region in Euclidland

Pumpkin π is on a 3 foot leash tied to an anchor in Euclidland. He can walk in any direction. The region he can access is a circle.



Your Turn: Leash Access Region in Gridland

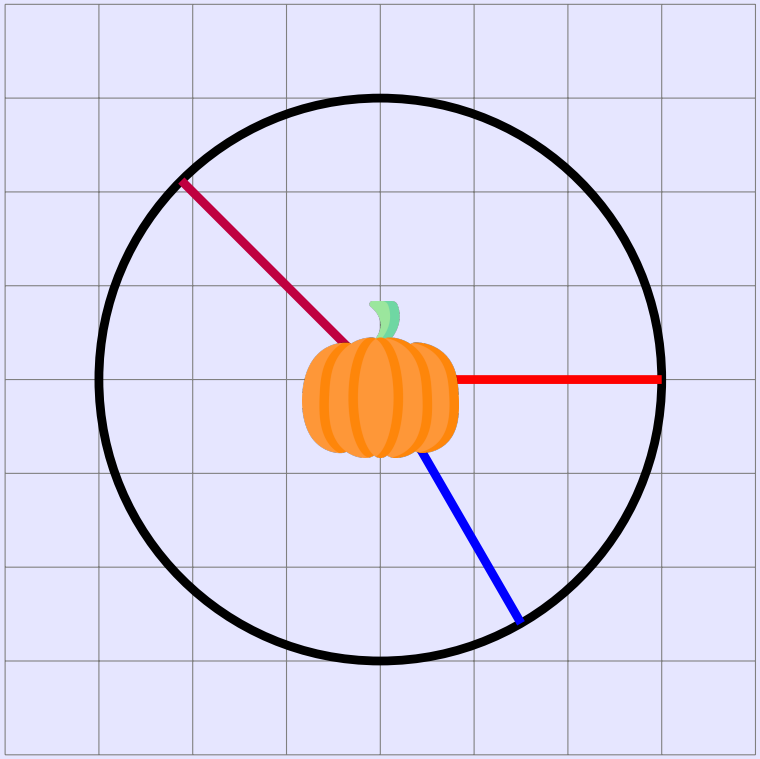
Pumpkin π's friend, Pumpkin 4, is on a leash tied to an anchor in Gridland. She cannot walk along diagonals. What shape is the region she can access?

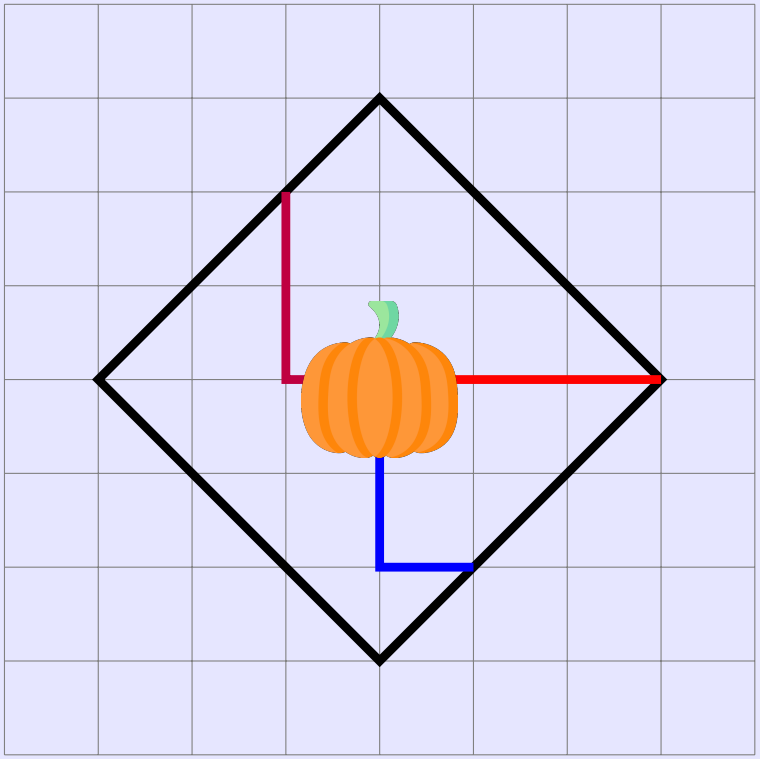
Draw on your grid paper!

- Pick an anchor point.
- Mark points that are 3 "blocks" away.

Leash Access Region in Euclidland vs. Gridland

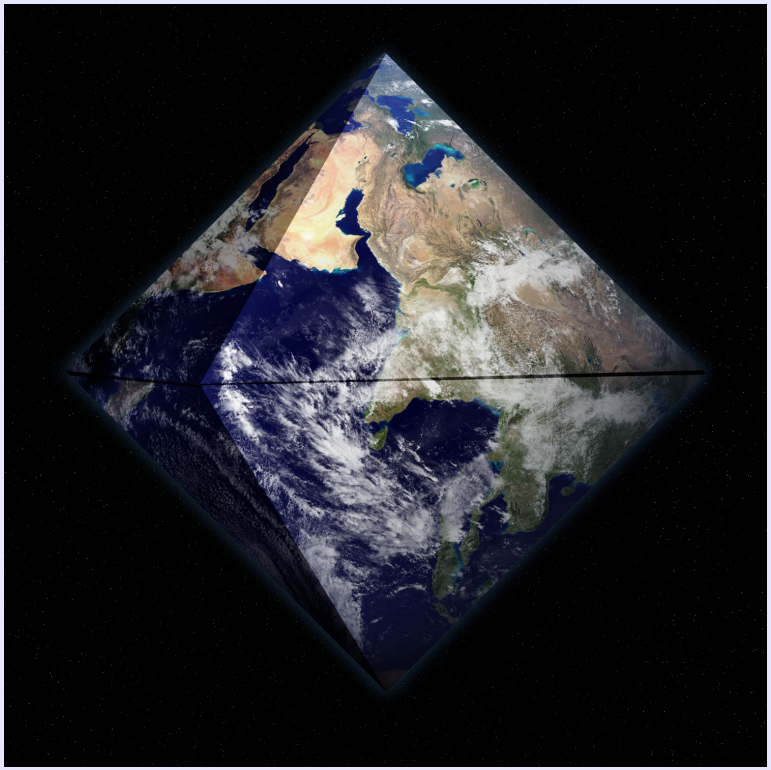
Pumpkin π is on a leash tied to an anchor in Euclidland. He can walk in any direction. The region he can access is a circle.
Pumpkin π's friend, Pumpkin 4, is on a leash tied to an anchor in Gridland. She cannot walk along diagonals. The region she can access is a square.
A circle in Euclidland becomes a square in Gridland.





What if Pumpkin π and Pumpkin 4 could fly?

Pumpkin π is on a leash tied to an anchor in Euclidland. He can fly in any direction. The region he can access is a sphere.
Pumpkin 4 is on a leash tied to an anchor in Gridland. She cannot fly along diagonals. The region she can access is an octahedron.
A sphere in Euclidland becomes an octahedron in Gridland.



Where did Pumpkin 4 get her middle name?

Pumpkin π's middle name:

$$\pi = \frac{\text{circumference of circle}}{\text{diameter of circle}} = \frac{\text{circumference of circle}}{2 \cdot \text{radius of circle}}$$

Where did Pumpkin 4 get her middle name?

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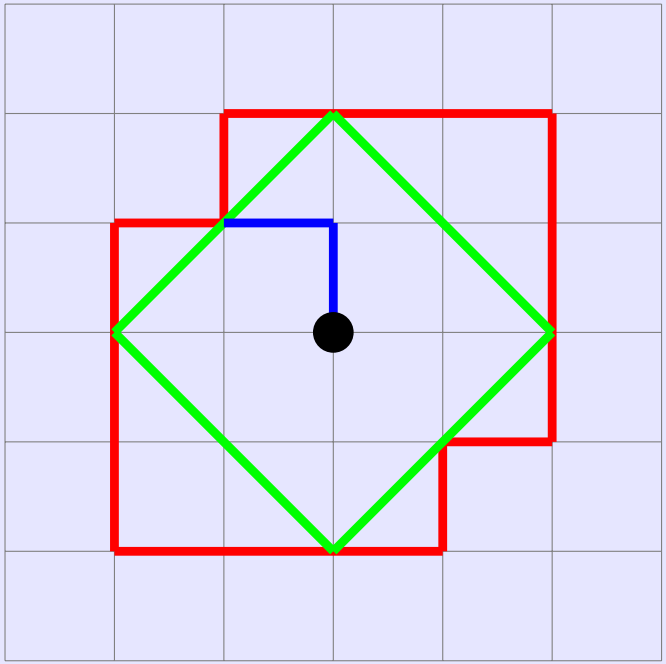
Recall: A circle in Euclidland becomes a square in Gridland.

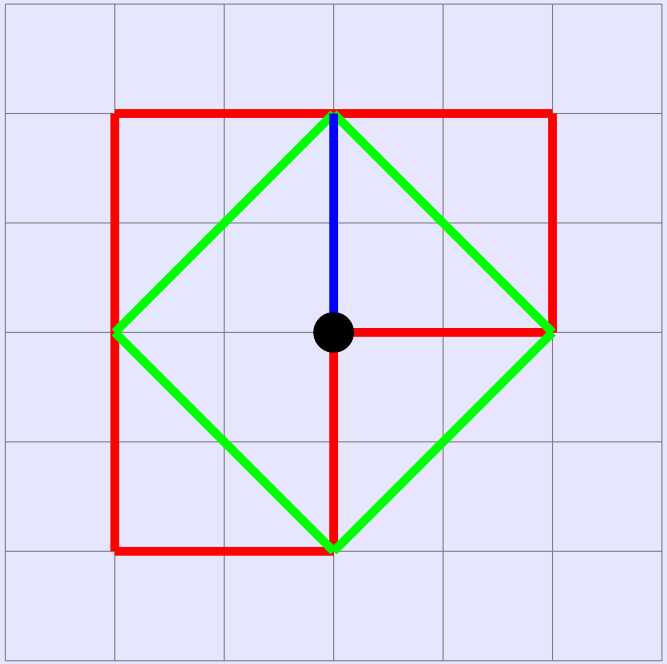
Pumpkin π's middle name:

$$\pi = \frac{\text{circumference of circle}}{\text{diameter of circle}} = \frac{\text{circumference of circle}}{2 \cdot \text{radius of circle}}$$

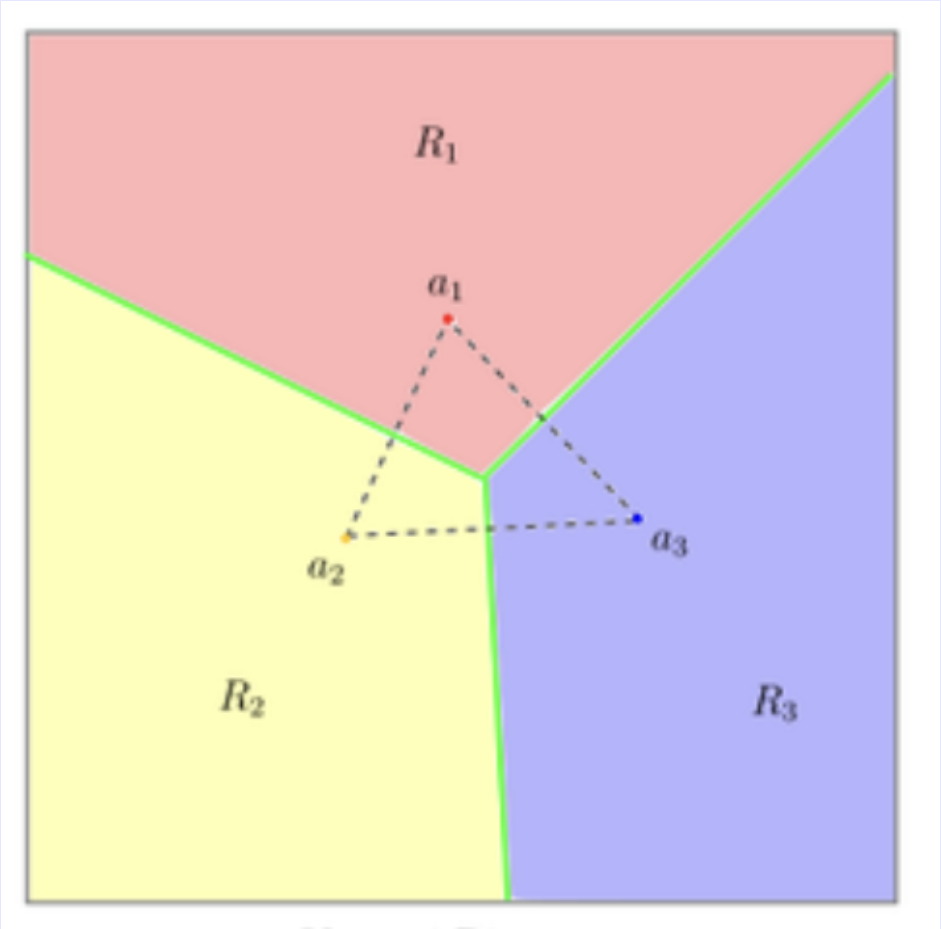
Pumpkin 4's middle name:

$$4 = \frac{\text{Gridland perimeter of square}}{2 \cdot \text{Gridland radius of square}}$$



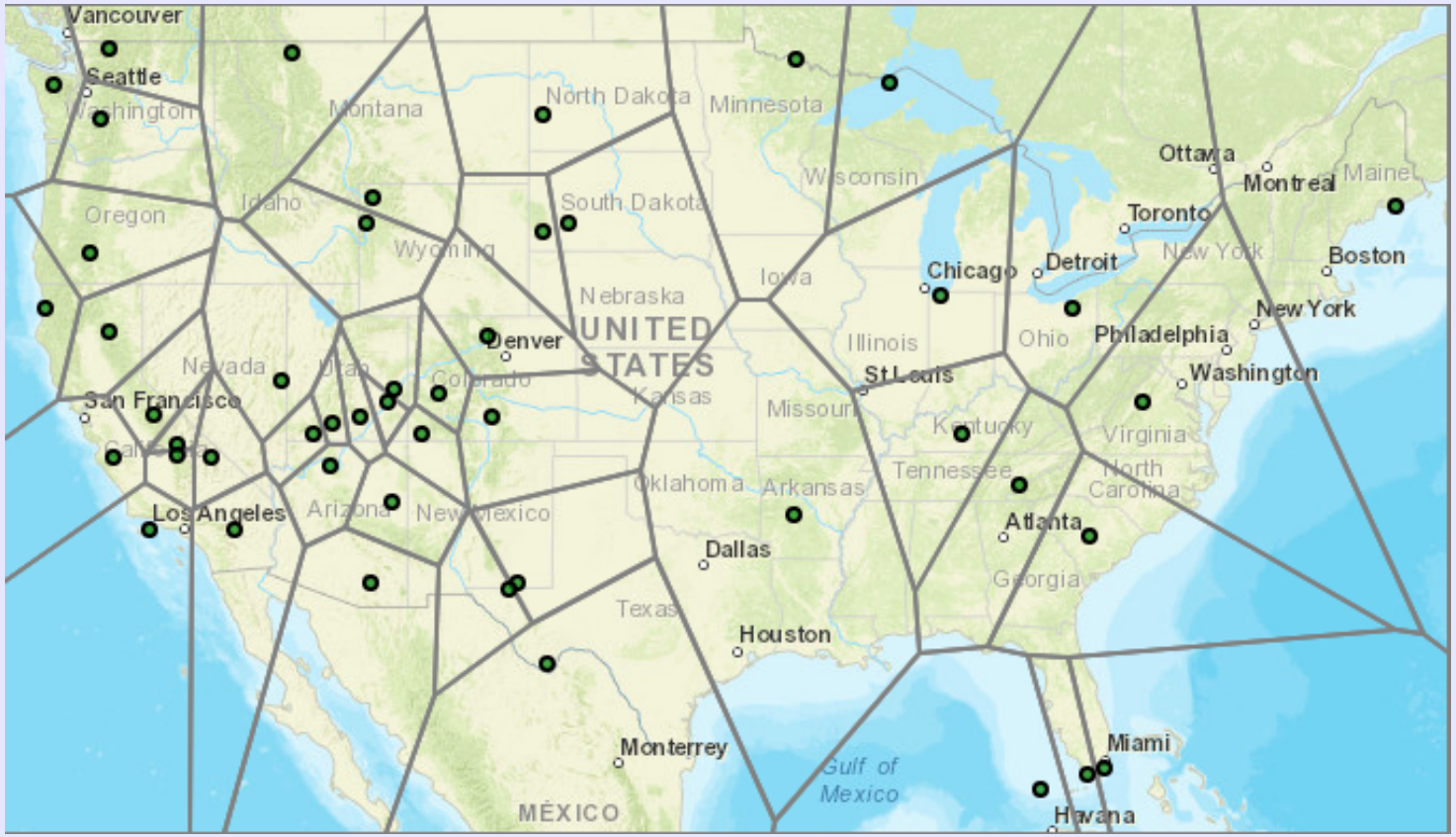


Voronoi Cells in Euclidland



Voronoi Cells in Euclidland

This map is divided into regions determined by where the nearest national park is.



Voronoi Application: Catchment Zones for Schools

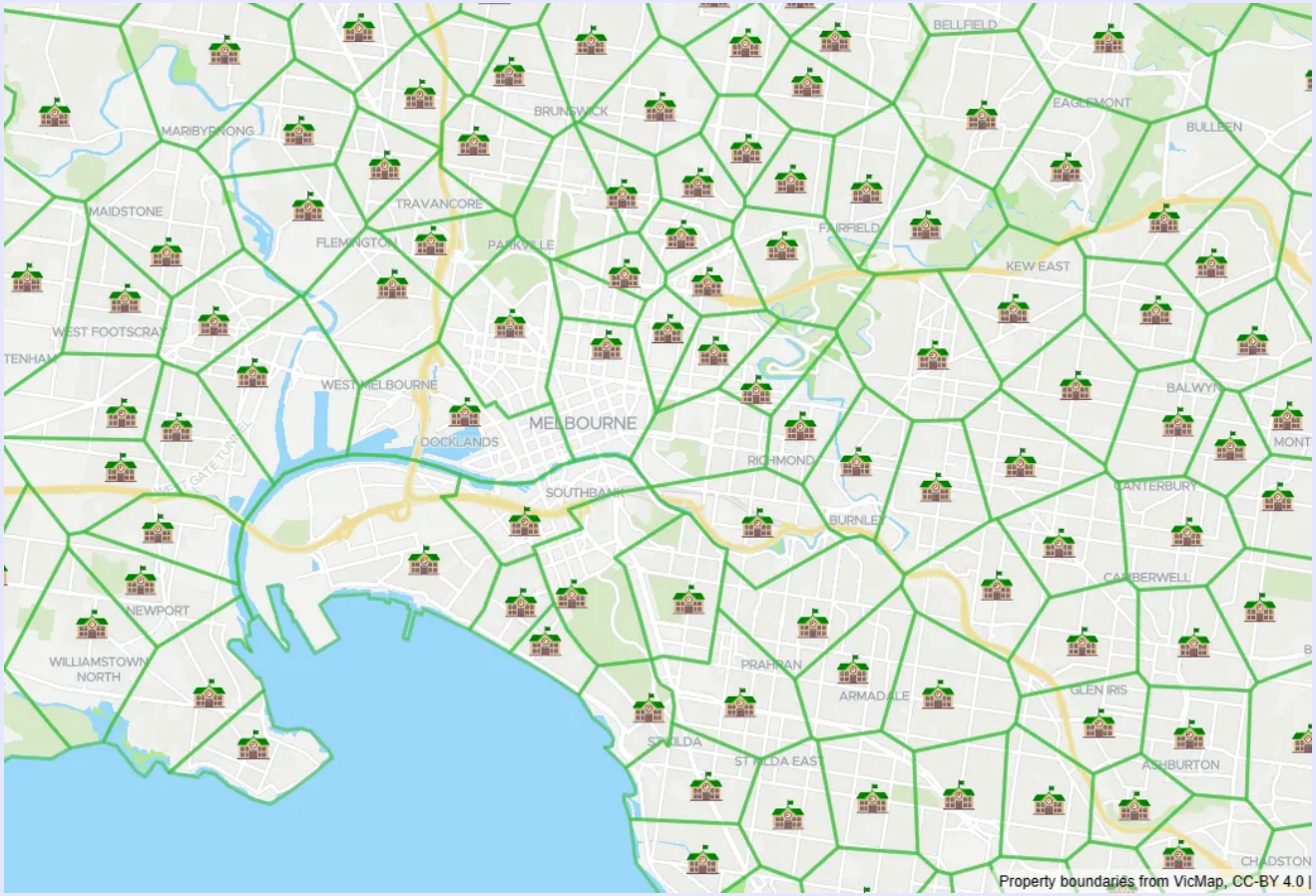
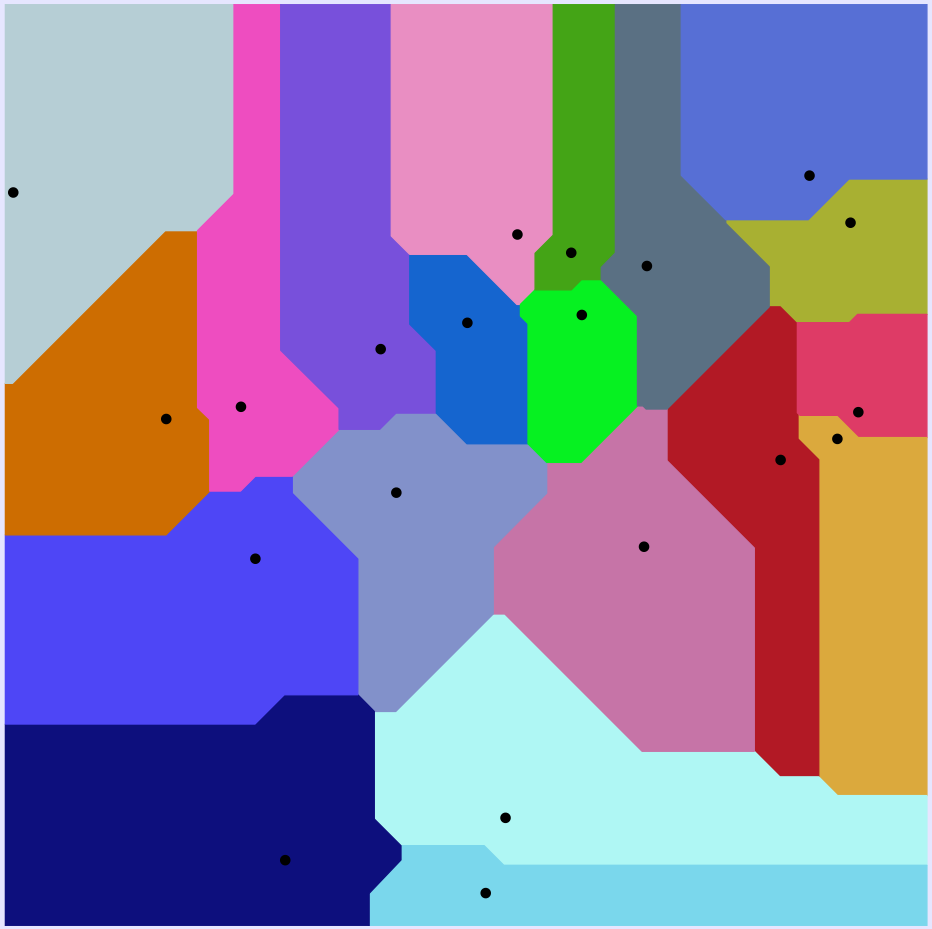


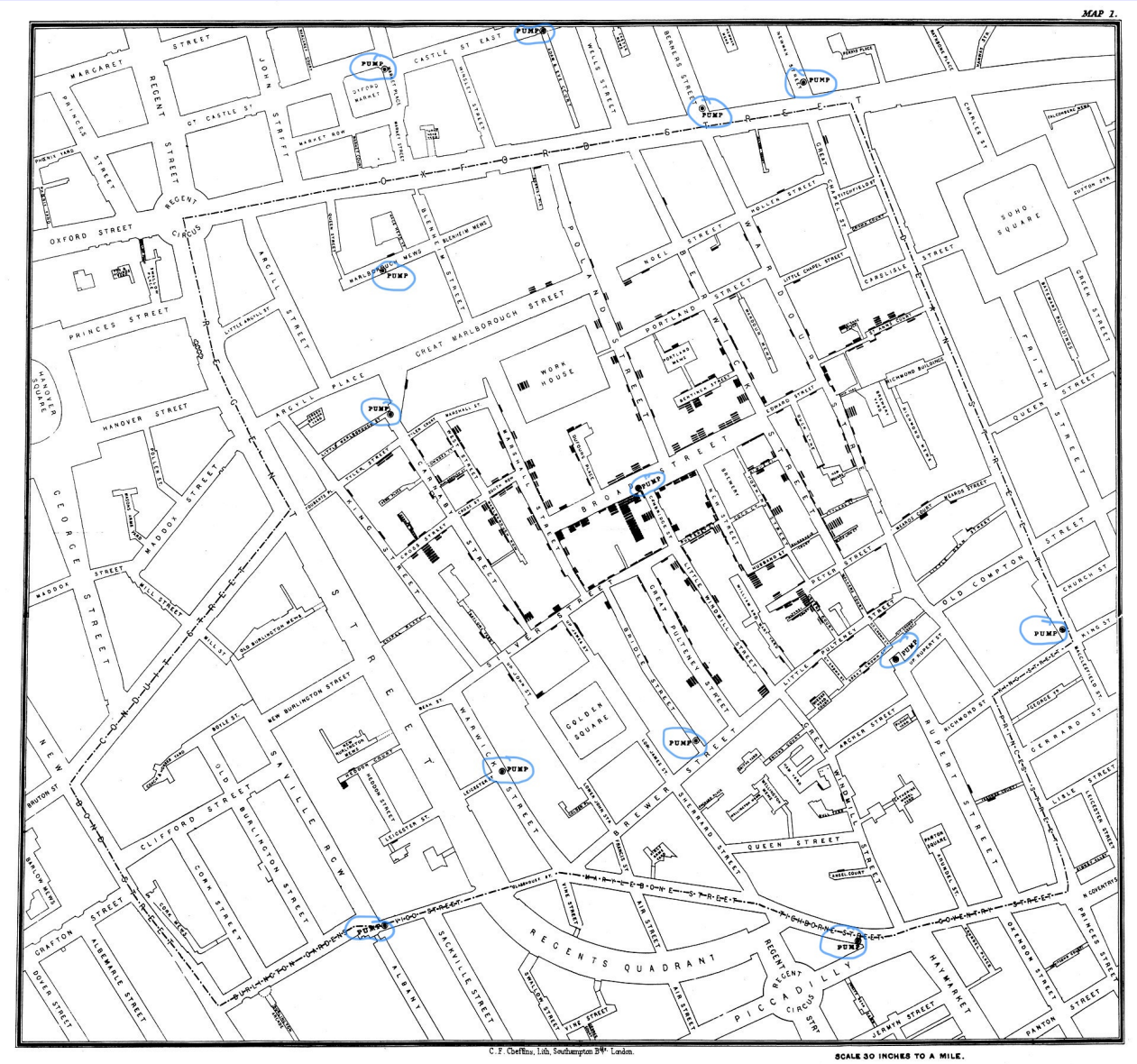
Figure: School catchment zones in Victoria, Australia

Voronoi Cells in Gridland

This image is divided into regions determined by where the nearest dot is, where nearest is defined using grid paths.



Voronoi Application: 1854 Cholera Outbreak in London

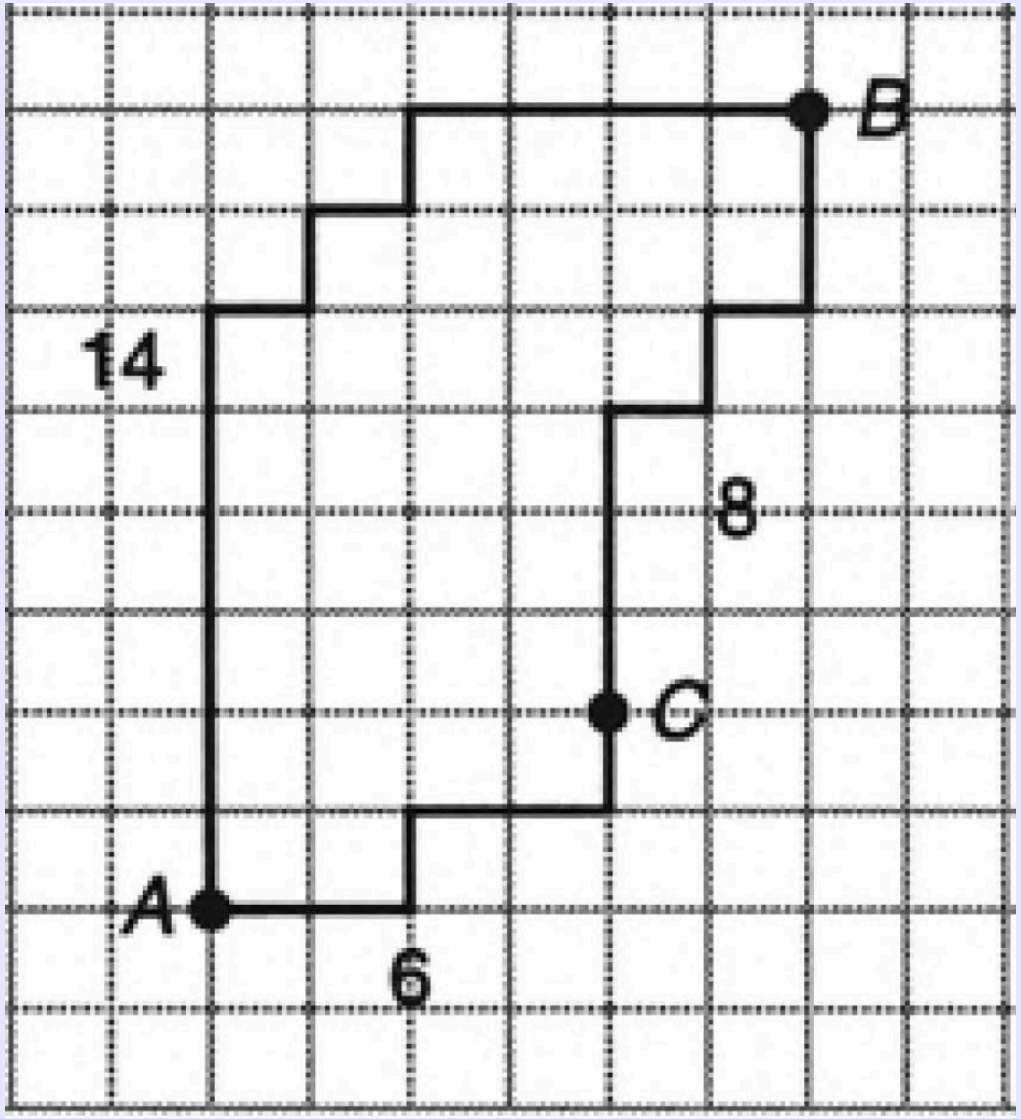


Shapes in Gridland

Earth	**Gridland**
circle	square
sphere	octahedron
straight line	grid path
triangle	?

Gridland Triangles

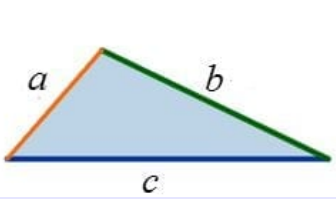
The sides of a Gridland triangle are grid paths.



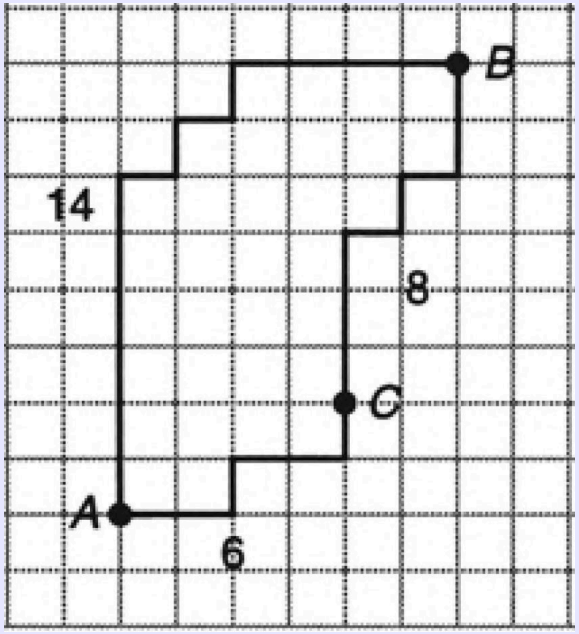
Triangle Inequality

Triangle Inequality: The sum of the lengths of any two sides of a triangle is greater than the length of the third. Relatedly, there is a unique shortest path between two points.

Not true in Gridland!

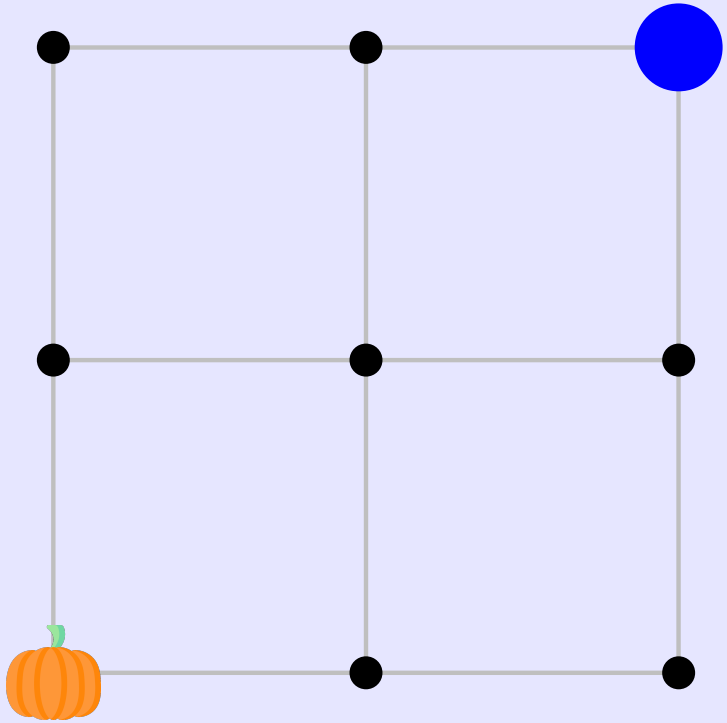


$$a+b>c$$
$$a+c>b$$
$$b+c>a$$

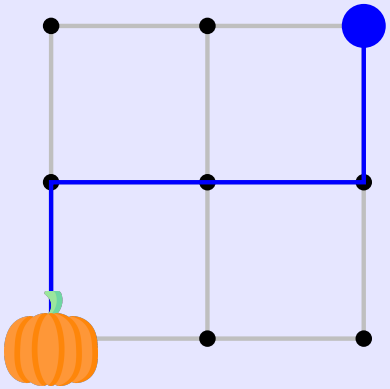
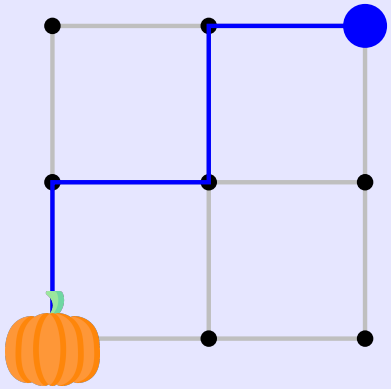
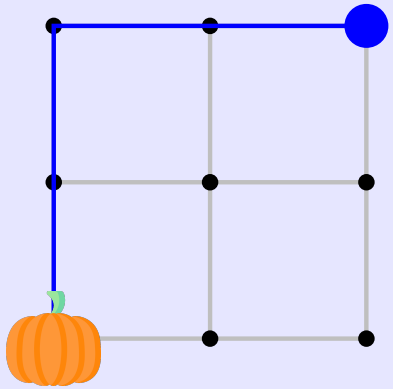
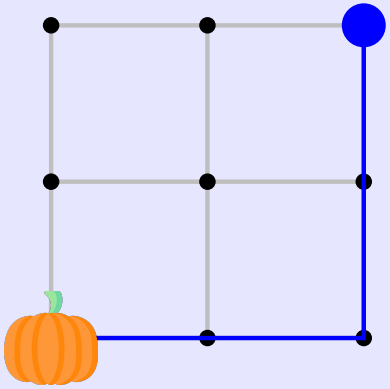
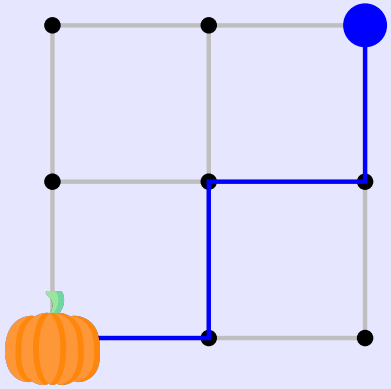
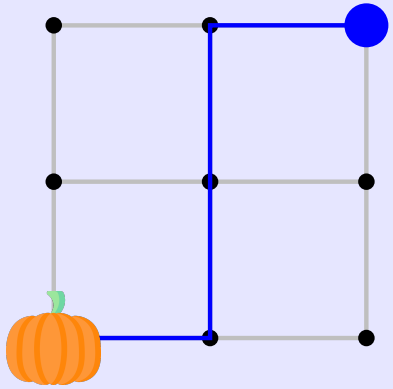


Your Turn: How many shortest paths are there between two points in Gridland?

Pumpkin 4 is playing fetch in Gridland. Her ball is 2 blocks north and 2 blocks east of her current location. She must walk along the grid. She does not want to take any extra steps. How many paths can she take to her ball?

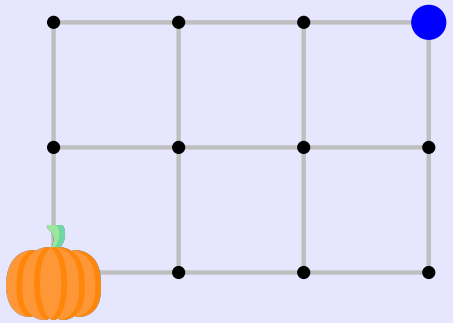


How many shortest paths are there between two points in Gridland?

Your Turn: How many shortest paths are there between two points in Gridland?

Now Pumpkin 4's ball is 2 blocks north and 3 blocks east of her. How many shortest grid paths are there?



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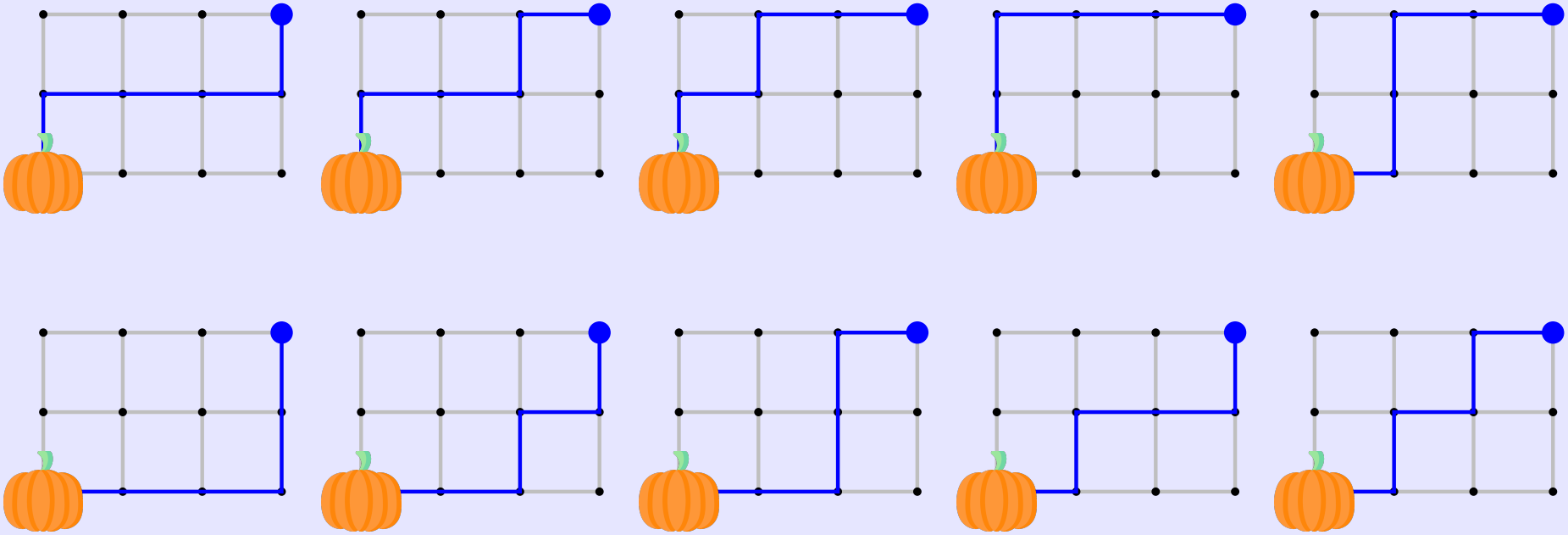
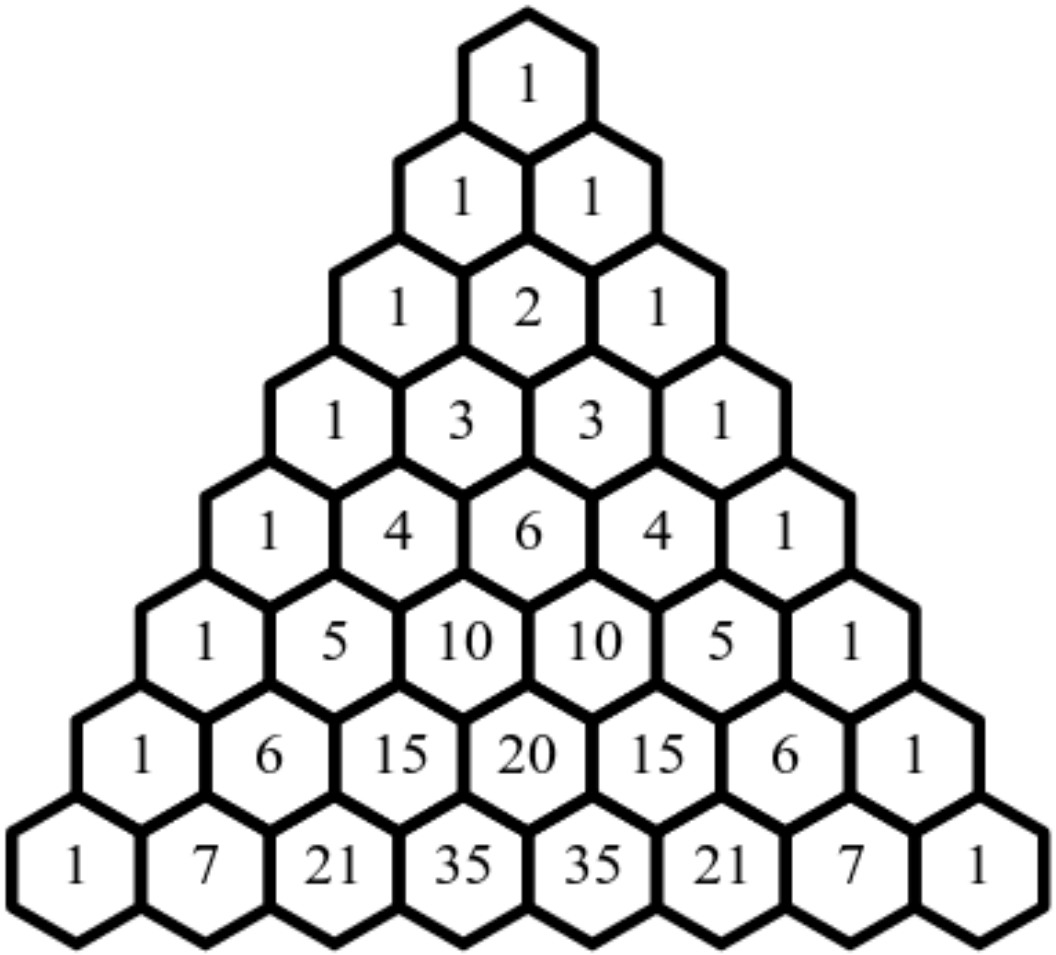
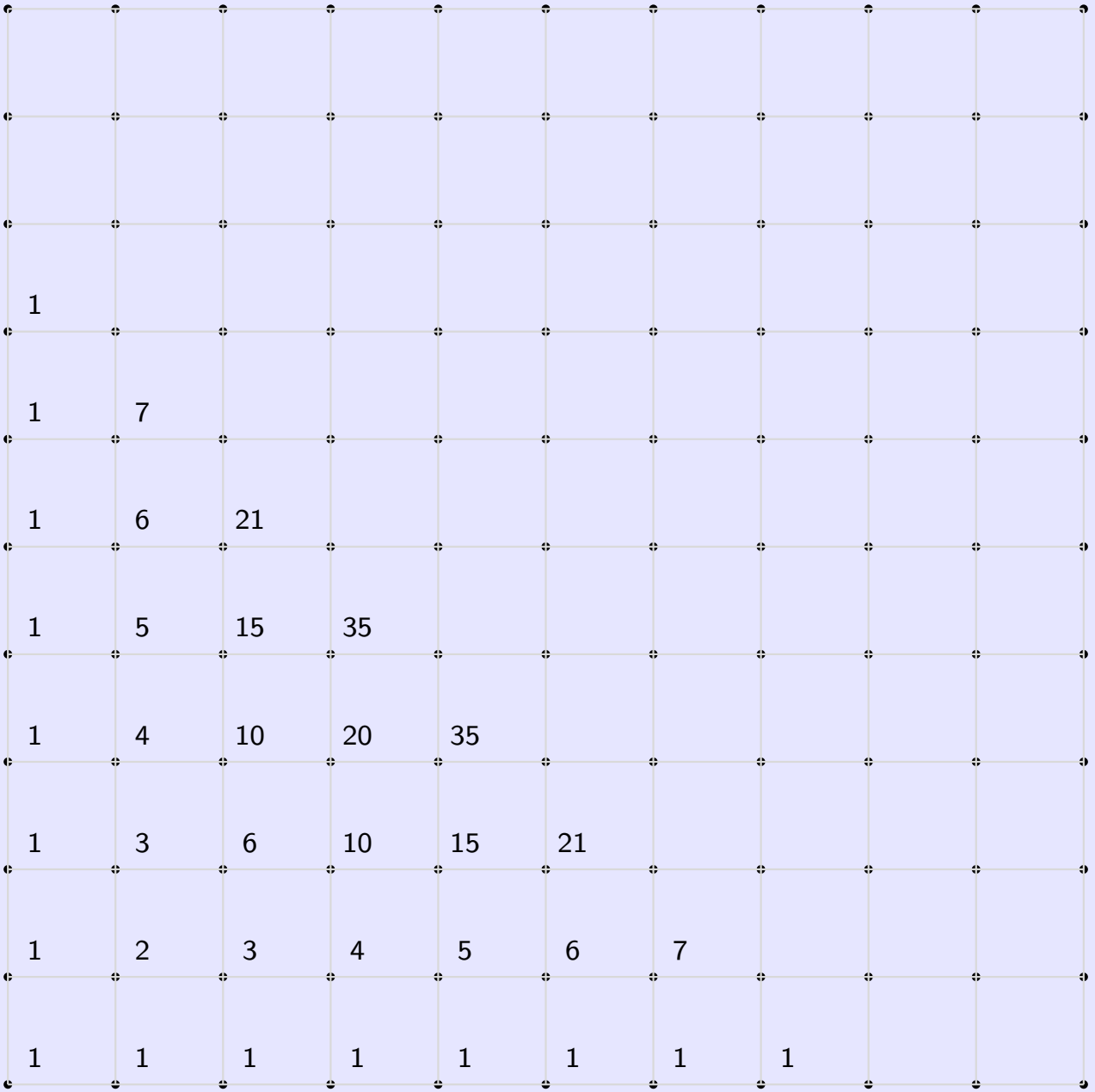


Pascal's Triangle

To get a number in the middle, add the two numbers immediately above it.



Pascal's Triangle Counts Gridland Paths



Challenge Question: How many shortest paths are there between two points in three-dimensional Gridland?

Pumpkin 4 can now fly! The ball is 3 blocks north, 4 blocks east, and 5 blocks up. How many shortest grid paths are there?

Ordering Distinct Letters

How many ways are there to order the letters ABC?

$$ABC, ACB, BAC, BCA, CAB, CBA$$

- There are three ways to choose what letter goes first.

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- So there are $3 \cdot 2 \cdot 1 = 6$ ways to order the letters.

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- So there are $3 \cdot 2 \cdot 1 = 6$ ways to order the letters.
- The notation $3!$ means $3 \cdot 2 \cdot 1$.

Ordering Letters with Repetition

How many ways are there to order the letters ABB?

$$ABB, BAB, BBA$$

- First, label one of the B's as $\tilde{B}$. Then there are $3! = 3 \cdot 2 \cdot 1 = 6$ ways.

Ordering Letters with Repetition

How many ways are there to order the letters ABB?

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- First, label one of the B's as $\tilde{B}$. Then there are $3! = 3 \cdot 2 \cdot 1 = 6$ ways.
- This counts each ordering of ABB twice because there are two $(2! = 2 \cdot 1)$ ways to order B and $\tilde{B}$.
  - $AB\tilde{B} = A\tilde{B}B$
  - $BA\tilde{B} = \tilde{B}AB$
  - $B\tilde{B}A = \tilde{B}BA$

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  - $AB\tilde{B} = A\tilde{B}B$
  - $BA\tilde{B} = \tilde{B}AB$
  - $B\tilde{B}A = \tilde{B}BA$
- So there are $\frac{3!}{2!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1} = 3$ ways to order the letters ABB.

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Example path: NNNEEEEUUUUU

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Example path: NNNEEEEUUUUU

There are $\frac{(3+4+5)!}{3!4!5!} = 27,720$ shortest grid paths.

Image Credits

- π vs pie comic: Grant Snider of Incidental Comics
- Earth sphere: https://climate.nasa.gov/news/2206/earth-from-space-15-amazing-things-in-15-years/
- Voronoi diagram of three points: Chaoyu Quan, https://www.researchgate.net/figure/The-left-figure-gives-the-Voronoi-diagram-of-three-points-a-1-a-2-a-3-in-R-2-R-1_fig12_328926172
- National Parks Voronoi: https://engaging-data.com/national-park-voronoi/
- School catchment zones: https://www.findmyschool.vic.gov.au/

Image Credits, continued

- Gridland Voronoi cells: https://en.wikipedia.org/wiki/Voronoi_diagram#/media/File:Manhattan_Voronoi_Diagram.svg
- Pumpkin: https://www.flaticon.com/free-icons/pumpkin"title="pumpkinicons"
- Pascal's triangle hexagons: https://markeggensperger.medium.com/benefits-of-pascals-triangle-number-of-ways-to-traverse-a-graph-3b6ea592aa7
- Gridland triangle: Martin Gardener, *The Last Recreations*, Figure 65: https://archive.org/details/springer_10.1007-978-0-387-30389-5/page/n163/mode/2up
- Triangle inequality theorem: https://www.cheenta.com/triangle-inequality-theorem-explanation/
- Gridland pi: Martin Gardener, *The Last Recreations*, https://archive.org/details/springer_10.1007-978-0-387-30389-5/page/n163/mode/2up

Thank you!