UW Math Circle

Week 20 – Math Auction

Auction Rules

In this mathematical auction, there are six problems, and you are looking for optimal solutions. That can mean achieving something as quickly as possible, or building as large an example of something as you can.

There will be three rounds where points are scored. For each problem and each round, the first verified submission by your team that beats the current record will score 1 point immediately. You can submit multiple solutions to each problem each round, but only the first verified solution will score a point. If your team holds the record at the end of the round, you will score 3 points. In the final round, 7 points will be awarded to the record holding team for each problem.

Note: Part of your submission will involve verifying to an instructor that your solution satisfies all the criteria.

Go Low or Go Home

At the end of each round, we will play a short game called "Go Low or Go Home" Each team picks a natural number (0 is considered a natural number for this game). Any team that picks the same number as another team is disqualified for that round. The qualifying team with the lowest number wins 4 points.


Patos
Sucia
Matia
Stuart
Waldron
Portage
Lummi
Orcas
Spieden
Sinclair
Shaw
Cypress
Blakely
Guemes
San Juan
Decatur
Lopez

1. The Washington State Department of Transportation has a bridge-building project in the San Juan Islands, and they've put you in charge! Your goal is to build a system of bridges between the islands to allow the following type of journey:

   (a) start on Orcas Island

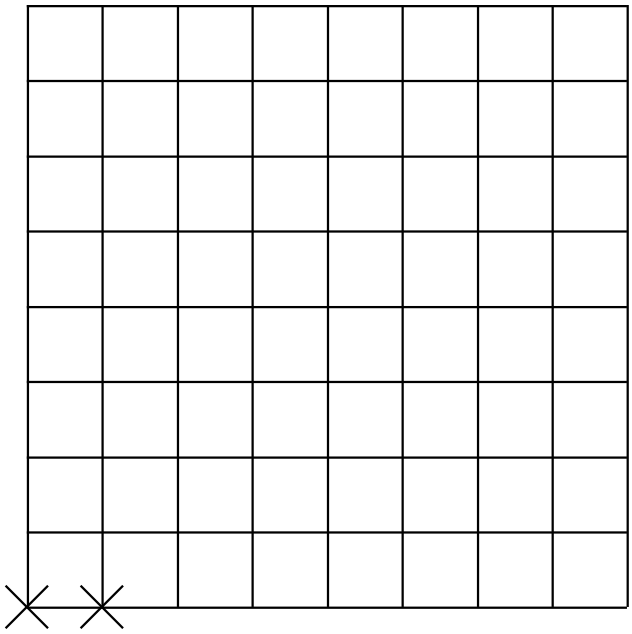
   (b) cross every bridge exactly once

   (c) visit each island **at most** 4 times

   (d) end on Orcas Island

   Your bridges do not have to be straight lines, but **two bridges cannot cross one another**. What is the largest number of bridges you can build?

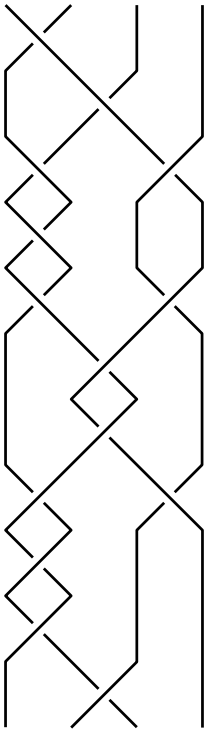
2. Recall a *linkage* is a network of bars attached at joints that can rotate. Consider the following linkage in the plane, which is in the form of a 8×8 grid. Every line segment is a bar, and every intersection is a joint. The two bottom left joints are pinned in place; they cannot move.

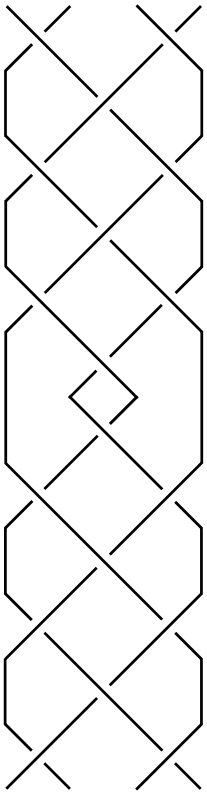


   What is the fewest number of bars you can add between existing joints to make the linkage rigid?

3. Recall two braids are equivalent if you can fix the ends in place and adjust the strands of one to get the other.

   You can modify the following braids in two ways: you can add an extra crossing at the end, or you can take an existing crossing and swap the over- and under-strands. Each of these operations counts as one move. What is the fewest number of moves needed to make each braid equivalent to the identity braid?





4. Happy St. Patrick's Day! A shamrock (three-leaved clover) takes up 6 squares on a chess board as shown in figure 2.

   (a) What is the largest number of shamrocks you can put on a 16×16 chess board such that no two overlap?

   (b) What is the smallest number of shamrocks you can put on a 16×16 chess board such that you cannot add any more without overlapping an existing shamrock?

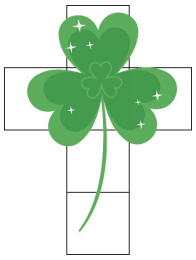
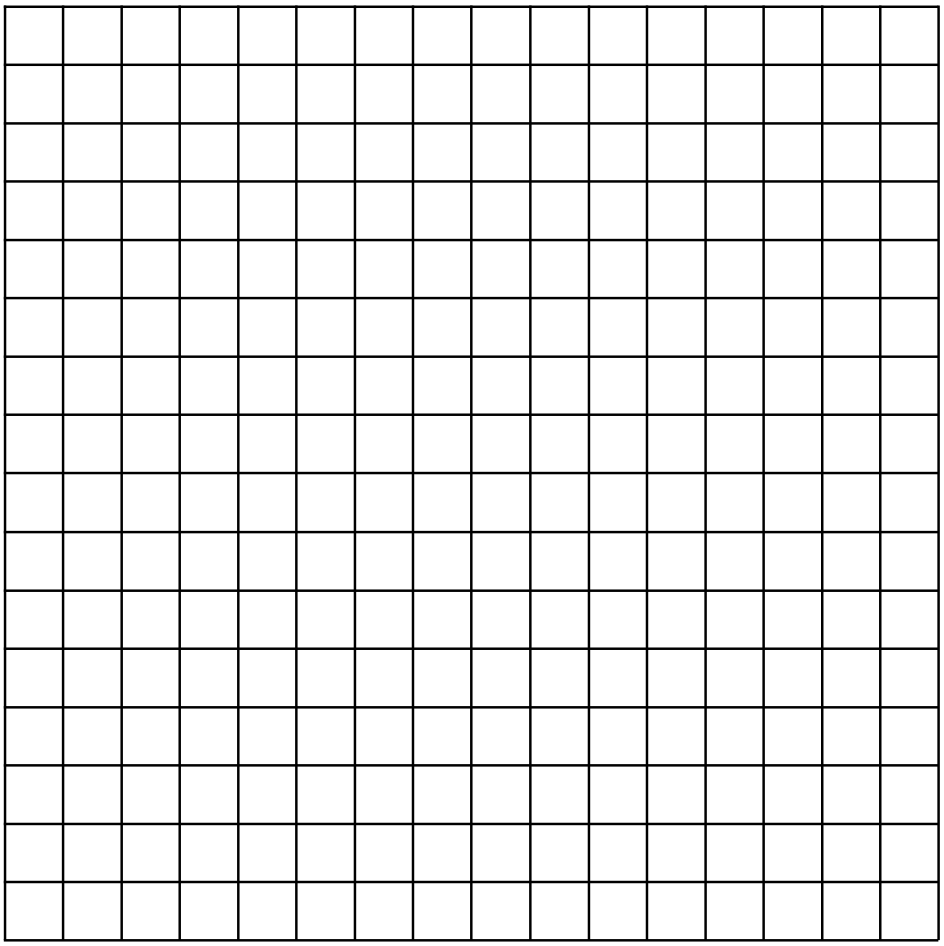


Figure 2: A shamrock



5. The Collatz function is defined as follows on natural numbers:

$$f(n) := \begin{cases} n/2 & n \text{ is even} \\ 3n+1 & n \text{ is odd} \end{cases}$$

It is conjectured (but not known!) that by repeatedly applying this function, one always arrives at the number 1. For example, if we start with 5, then $f(5) = 16, f(16) = 8, f(8) = 4, f(4) = 2, f(2) = 1$, and $f(1) = 4$, so we cycle $1 \to 4 \to 2 \to 1$ after reaching 1. The greatest number that appeared in the above "Collatz sequence" was 16. If $n \leq 1000$, what is the greatest number than can appear in a Collatz sequence?

6. The figures below show arrangements of 2×1 rectangles, called *dominoes*. You are allowed to perform the following moves:

   (a) rotate two adjacent dominoes that are aligned the same way

   (b) rotate a single domino with a gap on its long side

   (c) slide a domino one unit vertically or horizontally

   These are illustrated in figure 3.

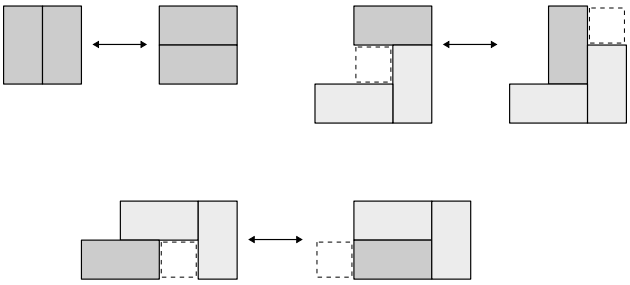


Figure 3: the three permitted domino moves

What is the fewest moves needed to turn the domino configuration on the left into the one on the right?