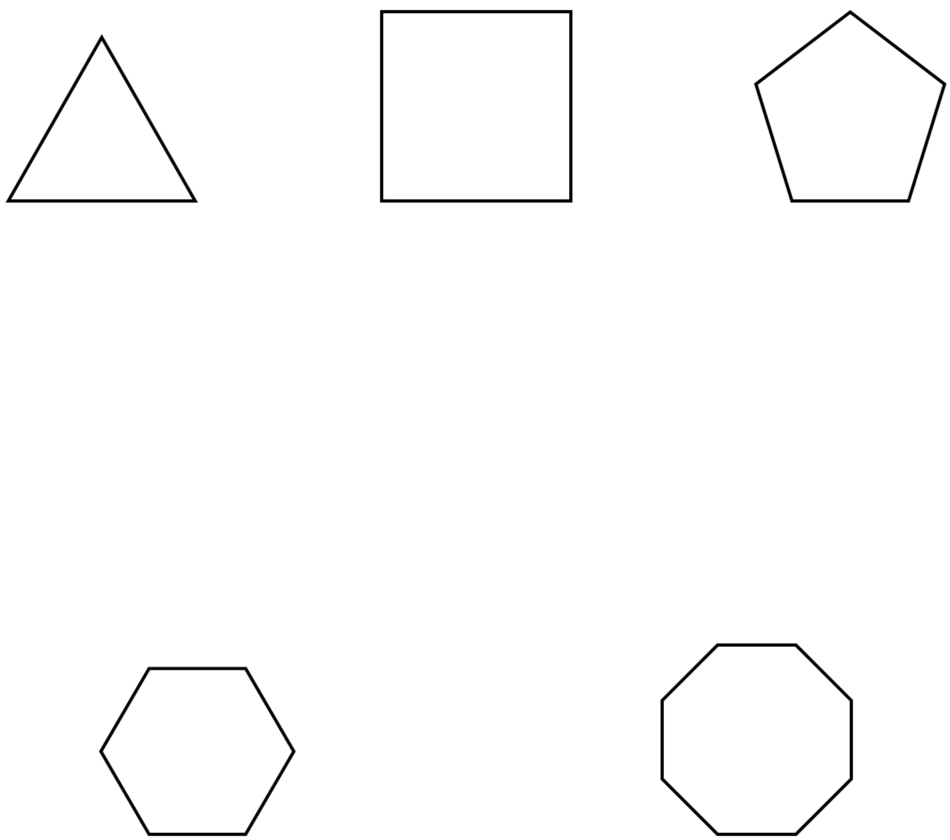
UW Math Circle
Week 18

Tilings with Regular Polygons

A *tiling* (or *tesselation*) of the plane is a way of covering the plane with shapes without any gaps or overlapping edges. A *regular tiling* is a tiling by a regular polygon where no corner of one tile meets the edge of another.

1. For each regular polygon below, identify whether or not it can make a regular tiling. Justify your answers.



2. A point where three or more tiles meet is called a *vertex* of the tiling. For each polygon from 1 that makes a regular tiling, draw a picture of what the tiles look like at a vertex. How many polygons meet at the vertex?

3. What do you think must be true about the interior angles of a regular polygon for it to tile the plane?

4. Are there any other regular tilings? Justify your answer. *(Hint: An interior angle of a regular n-gon measures* $\frac{180(n-2)}{n}^{\circ}$*)*

Stop here. Request the next page from your instructor when your group is done.

Now, let's loosen the rules a little bit. In a *plane-vertex* tiling, we allow more than one type of regular polygon to be used, but still impose the condition that no corner of one tile meets the edge of another.

5. Using your paper polygons, can you come up with a plane vertex tiling using...
   (a) 2 of the polygons that create a regular tiling?
   (b) all 3 of the polygons that create a regular tiling?
   (c) at least one polygon that can't create a regular tiling?

6. Are there any of the paper polygons that you think can never be used in a plane vertex tiling? Explain your answer.

7. An *Archimedean* tiling uses more than one type of regular polygon and has the same arrangement at each vertex. Which of your tilings from the previous question are Archimedean?

8. Can you create both an Archimedean and a non-Archimedean tiling using the same types of regular polygons? Draw a picture!

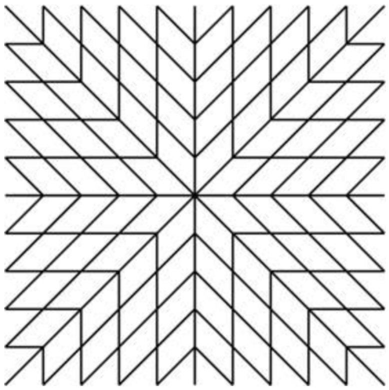
9. (*Challenge*) How many Archimedean tilings are possible? Can you find all of them?

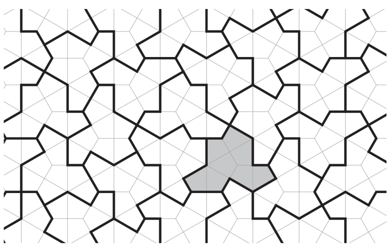
Stop here. Request the next page from your instructor when your group is done.

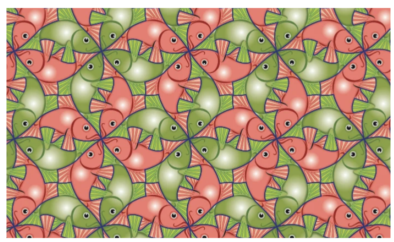
Periodic Tilings

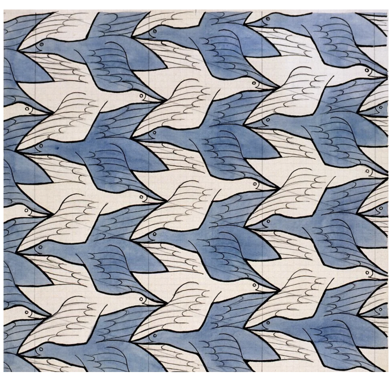
A tiling is *periodic* if it has 2-directions of translational symmetry, which means you can slide the pattern in 2 different directions and it still looks exactly the same. It is *aperiodic* if there is no translational symmetry.

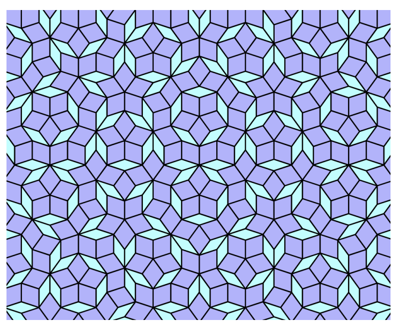
10. For the following tilings, are they periodic or aperiodic?









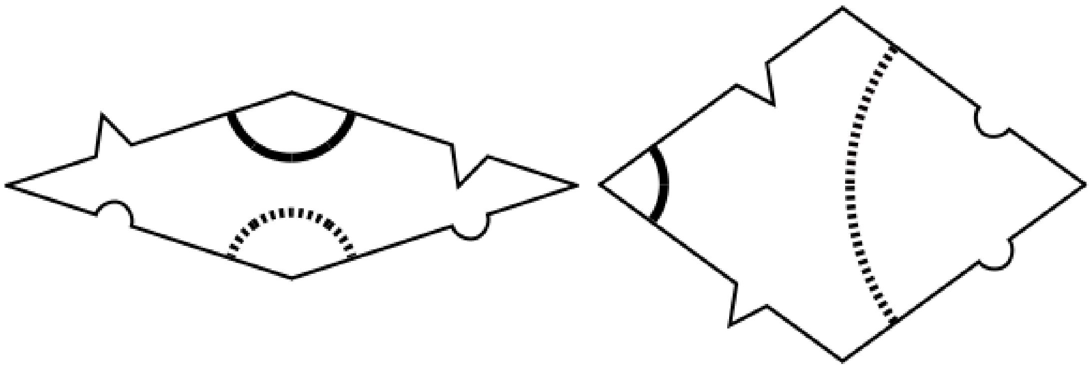


11. Can you make an aperiodic tiling of the plane using equal (but not necessarily equilateral) triangles? Draw a picture or explain why not!

12. Consider the following tile. Can you make both a periodic and an aperiodic tiling with it? Draw a picture.

13. Can you come up with a tile that can only tile the plane periodically? Draw a picture or explain why not!

14. Create a tiling using the following two tiles. Is it aperiodic or periodic? Draw a sketch! *(The paper tiles only have the lines; make sure the lines on each tile match in style and position to get the same tiling the notches would generate!)*



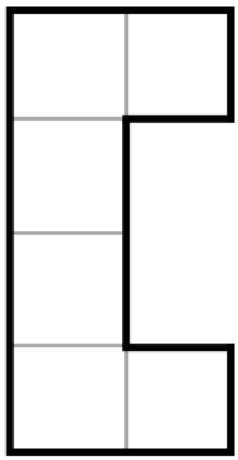
15. Using the same tiles, try to make a tiling that's the other type. What did you find?

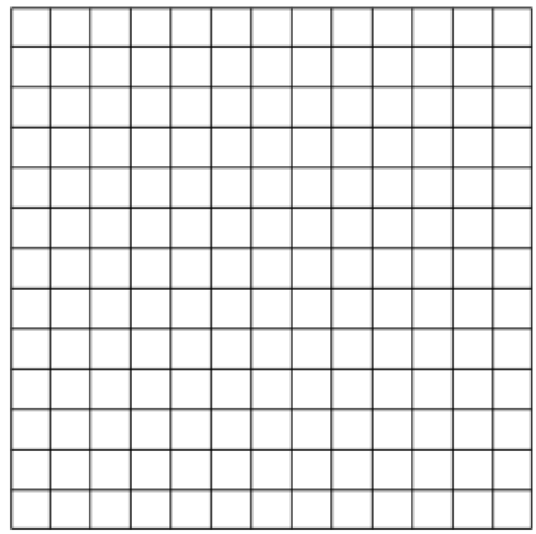
16. Do you think its possible to have a tile that can only tesselate aperiodically? (This is a hard question - mathematicians only answered it in 2024)

Stop here. Request the next page from your instructor when your group is done.

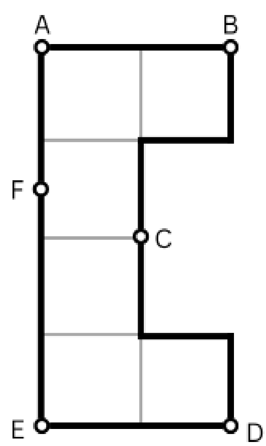
Conway Criterion

17. The following tile can tile the plane. Draw what the tiling looks like.





18. Here's the tile again, but now with some points labeled. What do you notice about segments AB and DE? The remaining edges BC, CD, EF, and FA all share the same symmetry condition. What is it? This is called the Conway criterion.



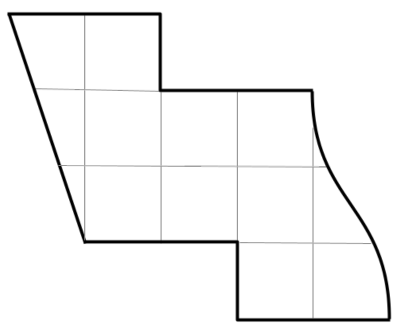
The Conway criterion says that a shape (with no holes) can be labeled with consecutive points A, B, C, D, E and F such that:

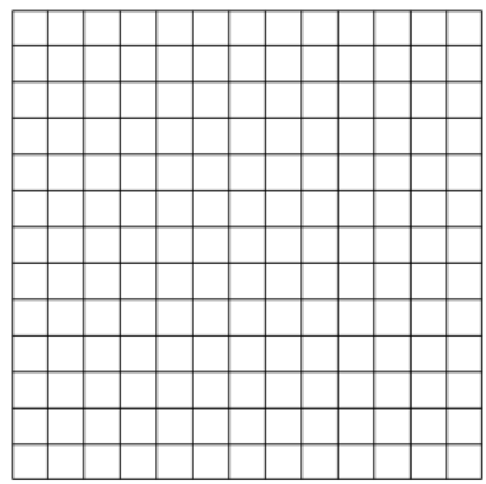
- you can slide AB to perfectly overlap DE
- BC, CD, EF, and FA are each equivalent to themselves after a 180° rotation.

Note that two points are allowed to be the same, but at least 3 must be distinct.

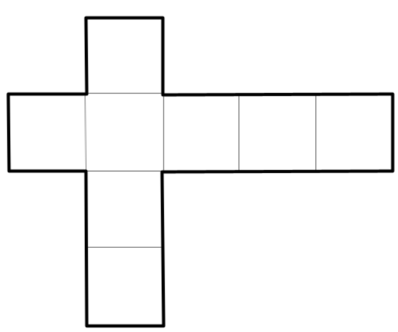
20. For the following tiles, identify if they satisfy the Conway criterion. Then decide if they can or cannot tile the plane. If they do tile the plane, draw a picture!

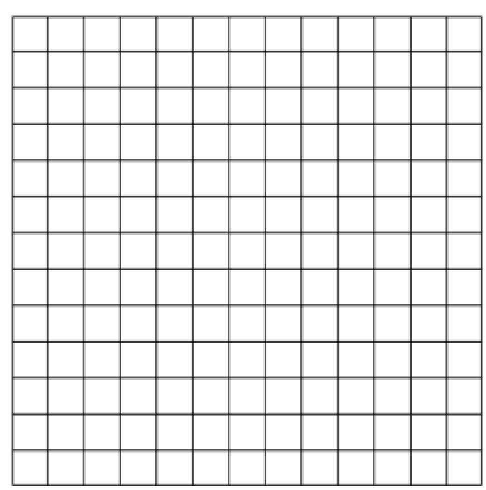
(a)



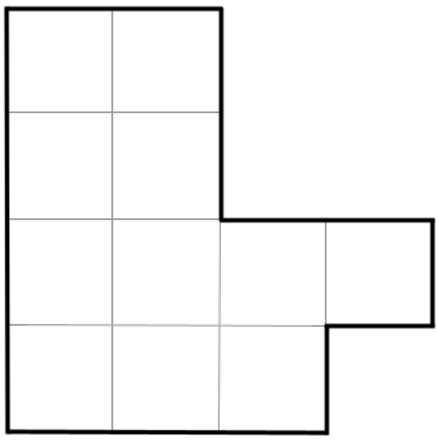


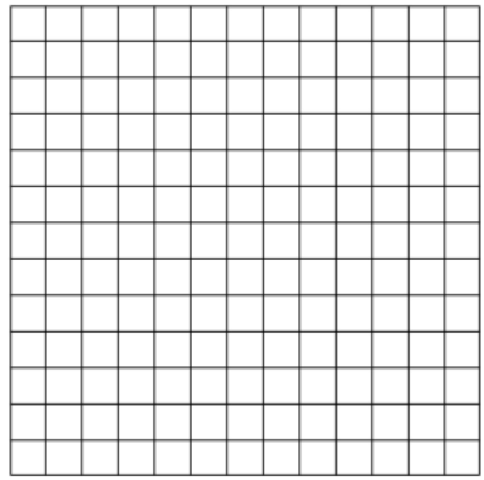
(b)



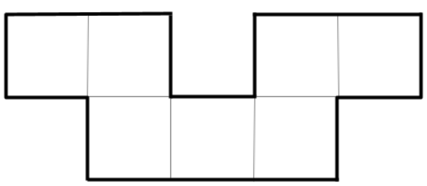


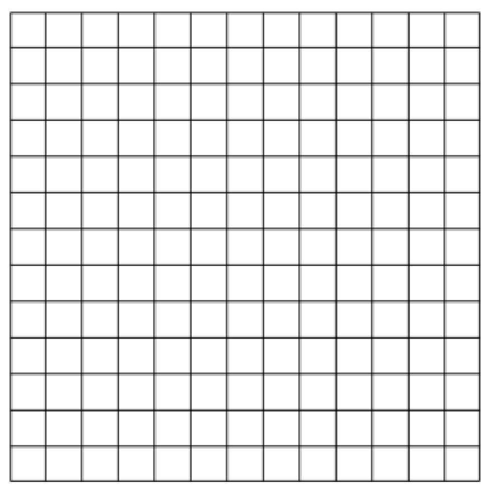
(c)





(d)





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Create your own!

21. Using the space on the handout, create your own tile that satisfies the criterion.

22. Switch with your partner, try to tile the plane with their tile using the grid. What did you find?

23. If the Conway criterion is met, does this guarantee that the plane can be tiled by the shape? If it's not met, does it mean the plane cannot be tiled by the shape?

24. Using the Conway criterion, explain why any quadrilateral (even non-convex ones!) can tile the plane.

Stop here. Request the next page from your instructor when your group is done.