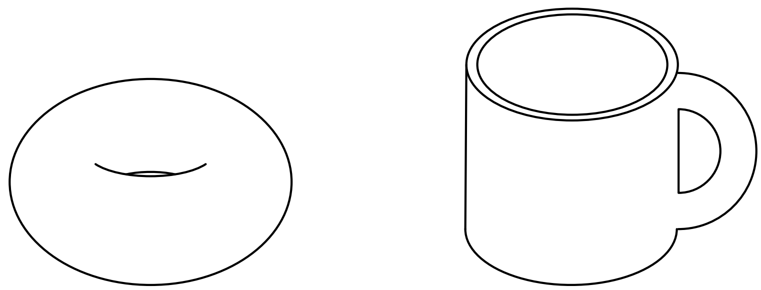
Equivalent shapes

1. Shapes in topology are squishy: we don't care about their sizes, angles, or proportions. Two shapes are *topologically equivalent* if you can turn one into the other by bending and stretching it.

   A classic math joke goes, "A topologist is someone who can't tell the difference between a doughnut and a coffee cup." **Use your modeling clay to explain the joke!**



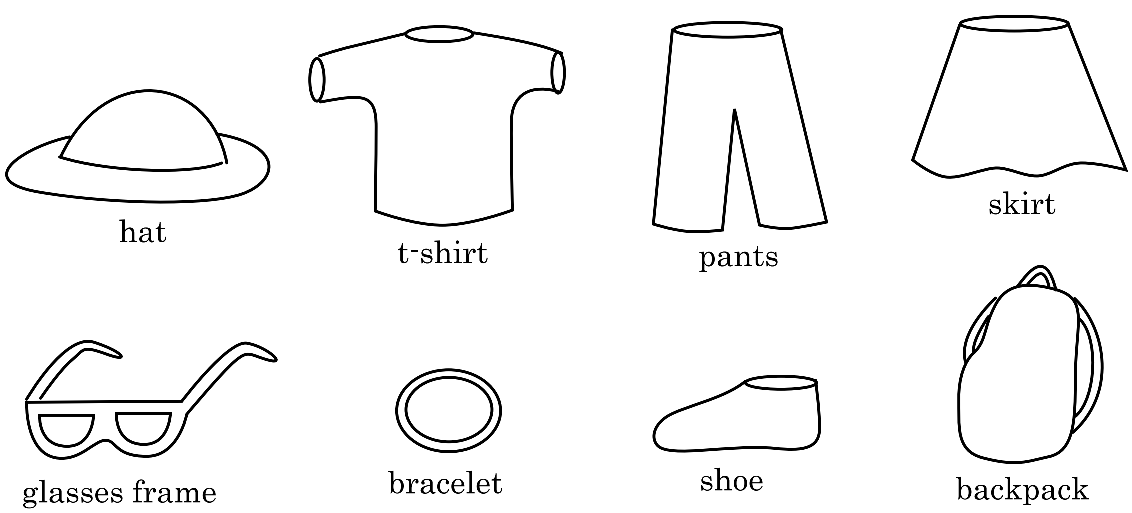
2. There are three rules to tell if shapes are equivalent:

   i. You can bend or stretch a shape as much as you want.

   ii. You *can't* break a shape apart or attach two parts that were previously separate.

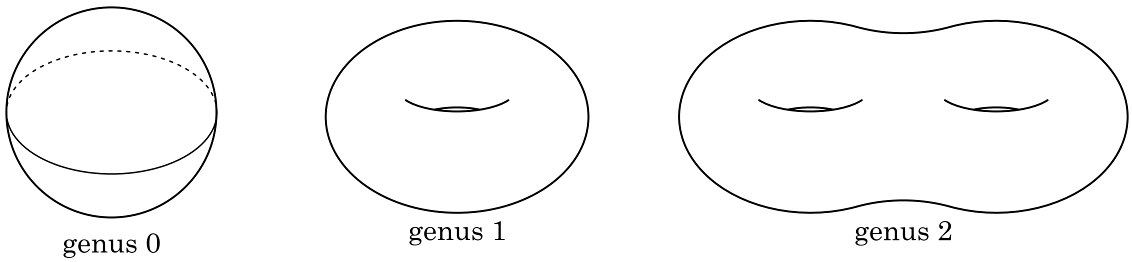
   iii. You *can't* poke a hole all the way through a shape or close up an existing hole.

   Using your modeling clay, **match the objects below that are equivalent!**

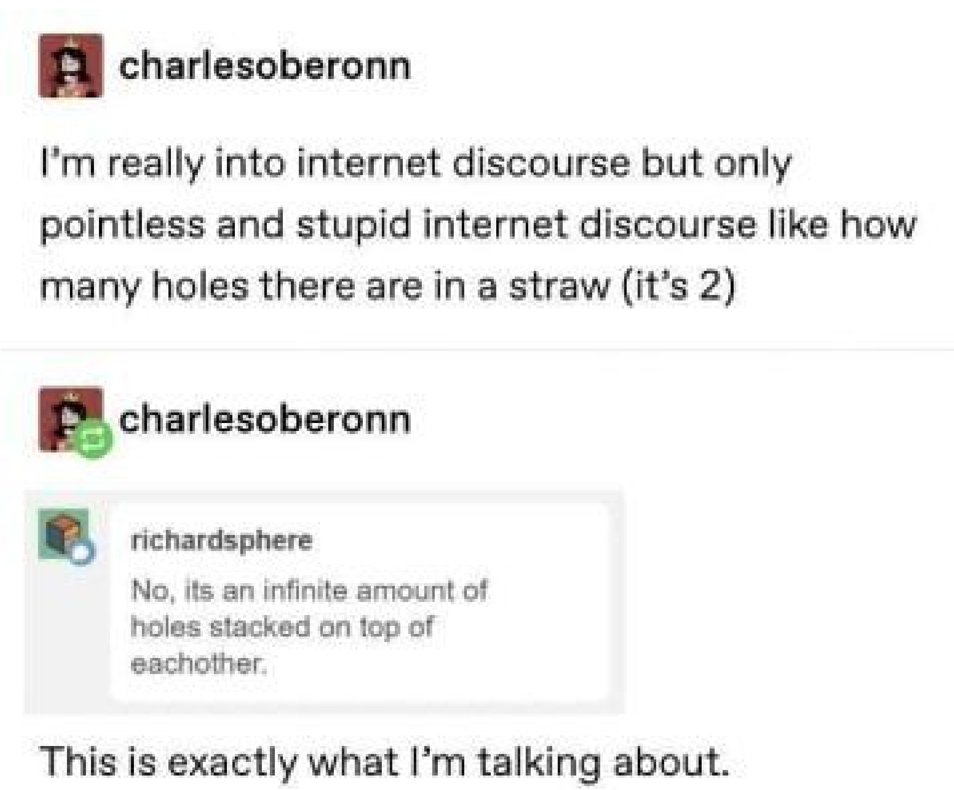


Genus

3. The number of holes in a shape is called its *genus*. Label the genus of each shape in Problem 2!



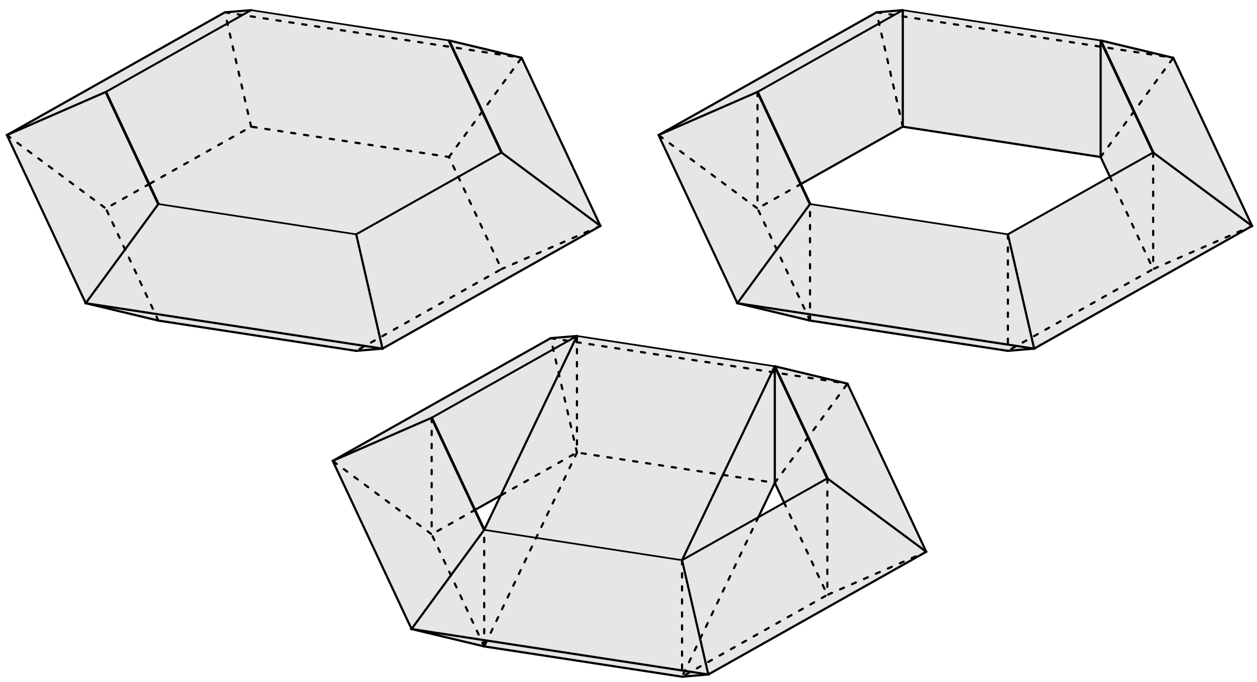
4. What do you think?



5. You might remember Euler's formula for polyhedra:

$$[\#\text{vertices}] - [\#\text{edges}] + [\#\text{faces}] = 2.$$

However, this formula doesn't apply to polyhedra with holes in them! Label the genus of each polyhedron below, then compute its $[\#\text{vertices}] - [\#\text{edges}] + [\#\text{faces}]$. This number is called the *Euler characteristic*.



6. Make a conjecture about how a polyhedron's Euler characteristic relates to its genus!

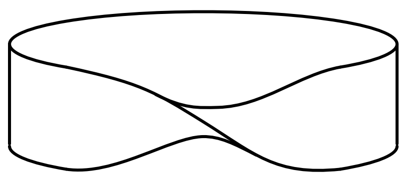
7. If two polyhedra have the same genus, will they always have the same Euler characteristic? Why or why not?

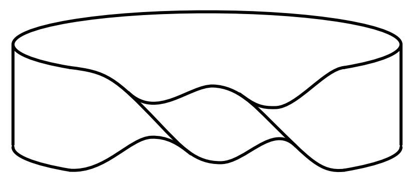
Möbius strips

1. Pick one person in your group to be *red* and one to be *blue*.

   Red: Take a red strip of paper and bring the ends together in a loop. Give one end a half-twist so the stars line up, then tape the ends together. This is called a *Möbius strip*!

   Blue: Take a blue strip of paper and bring the ends together in a loop. Give one end a *full* twist (two half-twists) so the stars line up, then tape the ends together. This is called a *double Möbius strip*!



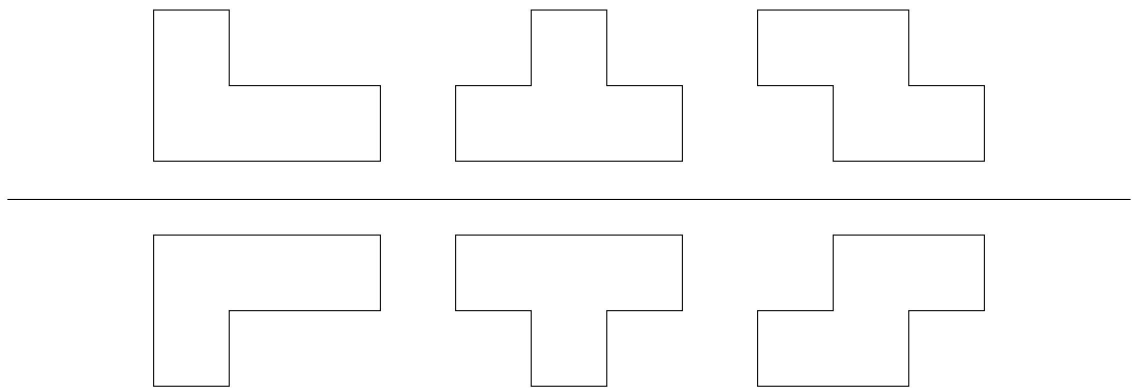


2. A sheet of paper usually has two sides. How many sides does a Möbius strip have? Hint: trace along the dotted line with your pencil until you get back to where you started!

3. If a sheet has two sides, it's called *orientable*; if it only has one, it's called *nonorientable*. Is a Möbius strip orientable or nonorientable? How about a double Möbius strip?

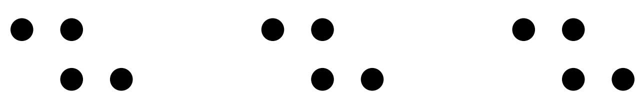
4. Set aside your Möbius strips—you don't need them for this problem.

   A 2D shape is called *chiral* (pronounced "kai-ral") if you *can't* shift or rotate it to look like its mirror image without picking it up. Which of these shapes are chiral?



5. The letter "Z" is chiral, so if it walks around a plain sheet of paper, it should never end up looking like an "S". Let's see what happens when it walks around a Möbius strip!

   Punch a "Z" into your red Möbius strip with a hole punch. Punch a second "Z" about an inch to its right, and continue all the way around. What happens?



6. How does what you saw in Problem 5 relate to orientability?

Cutting Möbius strips

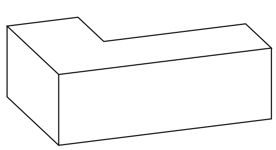
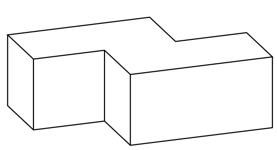
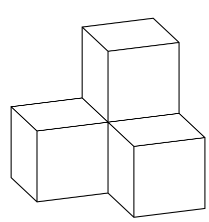
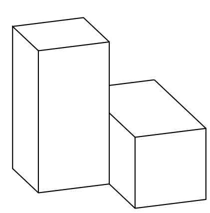
6. If you cut a Möbius strip in half, what do you expect the pieces will look like? Discuss with your partner, but **don't spoil it if you've done this before!**

7. Once you've discussed Problem 6, cut your red Möbius strip along the dotted line. Was your prediction correct? Why or why not?

8. If you cut a *double* Möbius strip in half, what do you expect the pieces will look like? Discuss with your partner!

9. Once you've discussed Problem 8, cut your blue double Möbius strip along the dotted line. Was your prediction correct? Why or why not?

10. *Bonus chirality problem!*—Since we live in three dimensions, it doesn't really matter if 2D shapes are chiral: we can always pick one up and flip it over. However, 3D shapes can also be chiral! Build each shape below along with its mirror image. **Which ones are chiral, and which aren't?**

Fundamental polygons

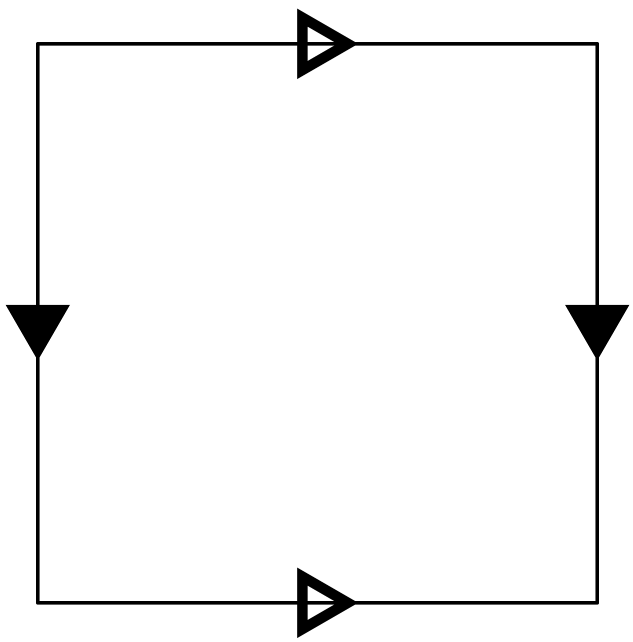
1. Look at the square below. The ▶'s mean we imagine the left side "wraps around" to the right: if you walk out the left, you'll walk back in the right (like Pac-Man). Similarly, the ▷'s mean the top "wraps around" to the bottom.

   This is called a *fundamental polygon*. What shape does it represent?

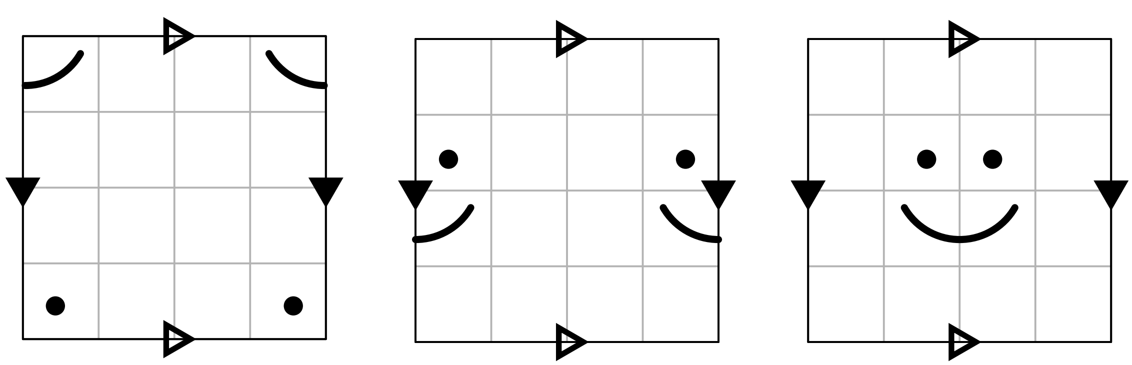
   i. Take one square, join the left and right sides so the ▶'s meet, and tape them together.

   ii. Join the top and bottom so the ▷'s meet, and tape them together.

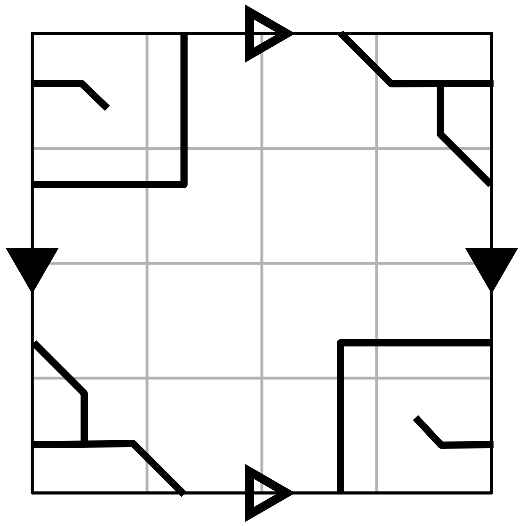
   iii. **Draw a picture** of the resulting shape (called a *torus*)!

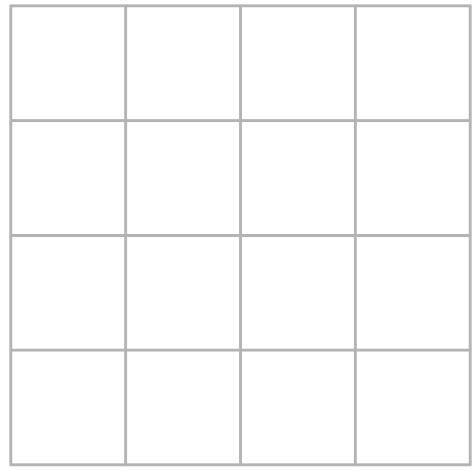


2. All three pictures below represent the same image on a torus. They *look* different, but only because we haven't wrapped them up yet.

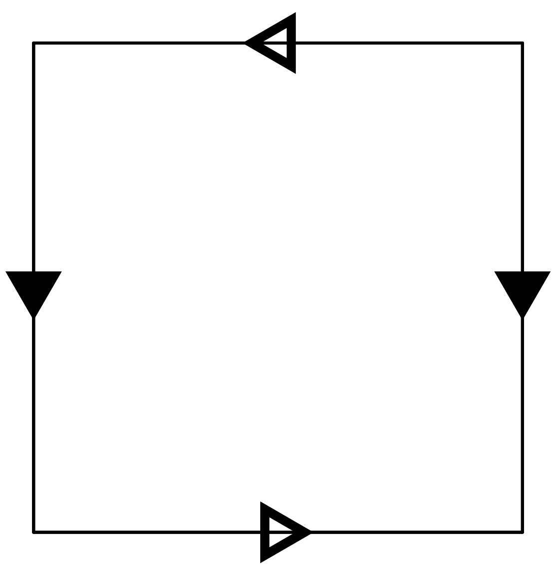


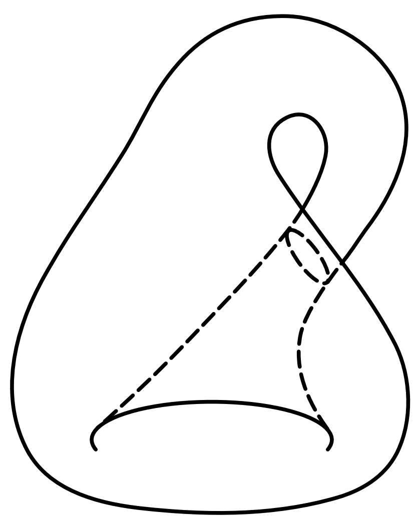
There's another picture on the fundamental polygon below. **Draw what the picture looks like connected up!**



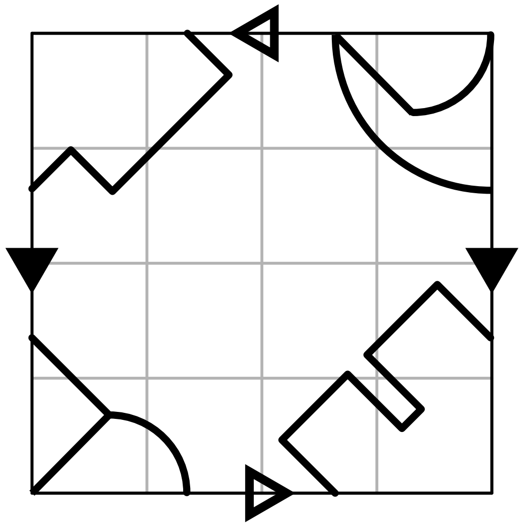


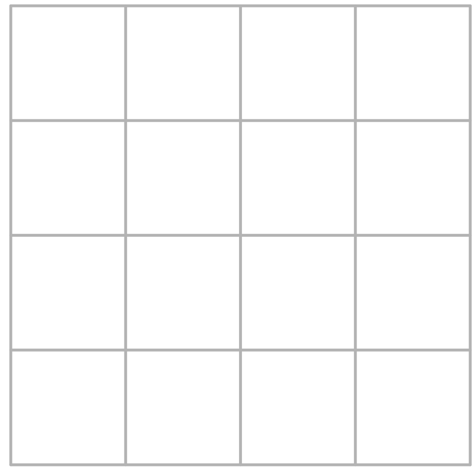
3. This fundamental polygon represents a *Klein bottle*. The opposite arrows on the top and bottom mean they're connected in *reverse*: if you walk out the top close to the left side, you'll walk back in the bottom close to the right side, and vice versa.



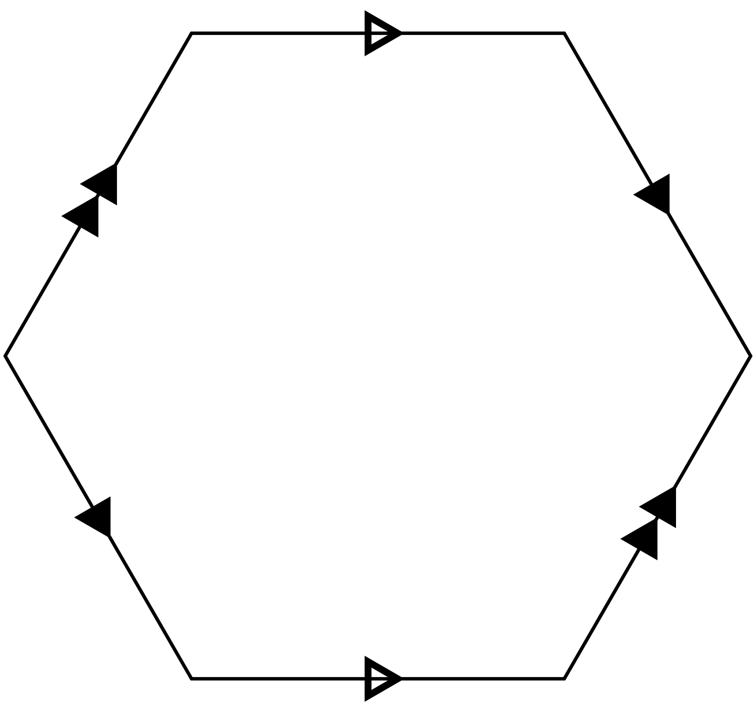


There's a picture on the fundamental polygon below. **Draw what the picture looks like connected up!**

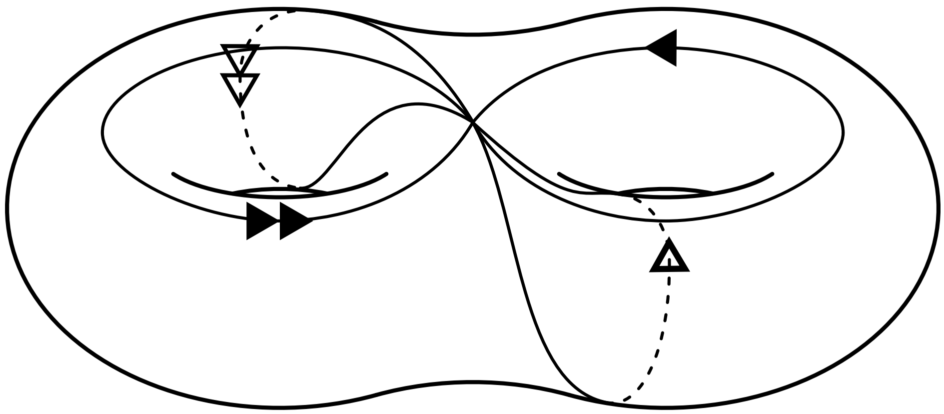




4. What shape does this fundamental polygon represent? Draw a picture!



5. Let's draw a fundamental polygon to represent the *double torus*!

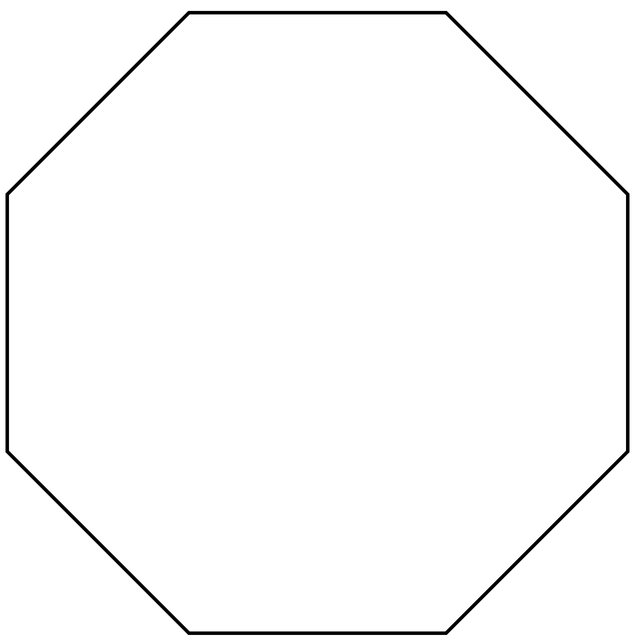


i. Build a double torus out of modeling clay. Draw the four lines and arrows shown.

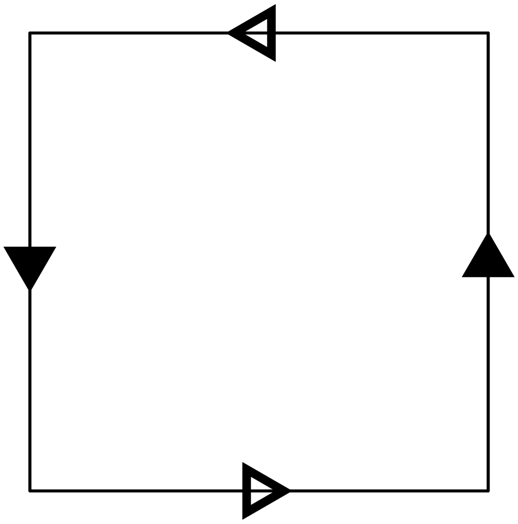
ii. Imagine you're standing on the double torus, and the lines are fences. You place your left hand on the fence and walk along it, then turn and walk along the next fence, and so on, until you're back where you started—**keeping your left hand on the fence and never crossing it**. Write down in order:

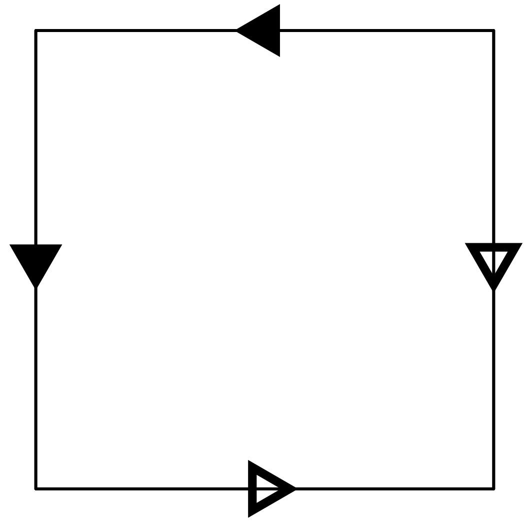
- Which arrow did you pass (►, ▷, ►►, or ▷▷)?
- Did it point the *same* or *opposite* direction you were walking?

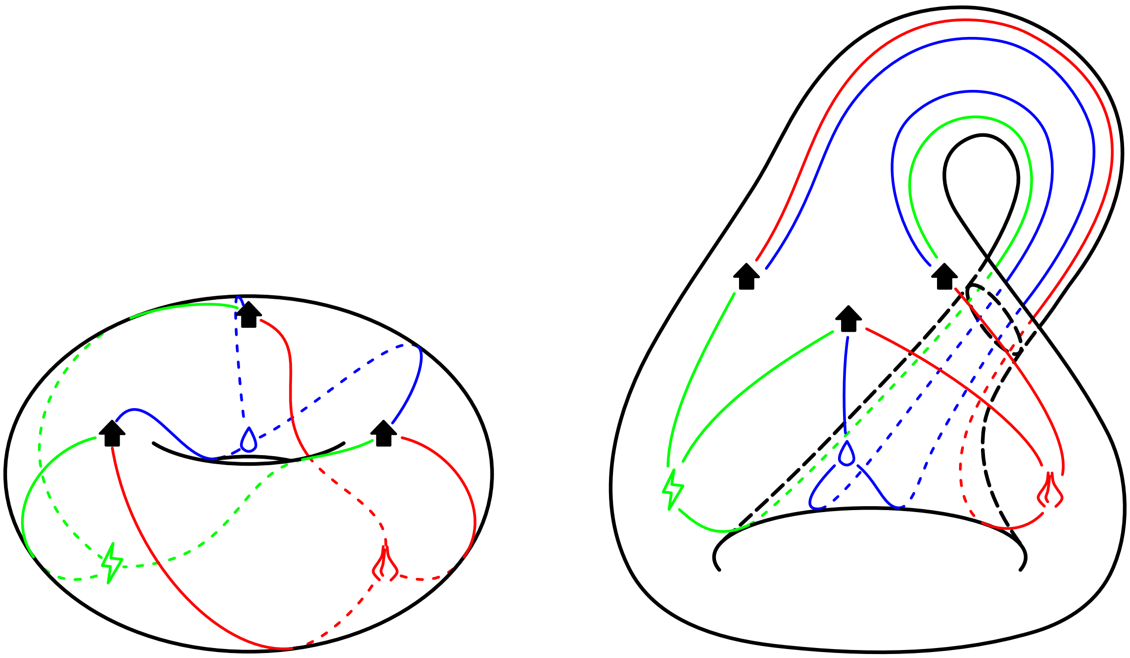
iii. If you walk around your fundamental polygon clockwise, it should be the same as walking around the fence! Going clockwise around the octagon below, mark arrows in the same order you recorded from step ii. Each type of arrow (e.g. ►) should appear twice!



6. *Challenge!*—These fundamental polygons each represent a shape that's very hard to draw (though you can try!). Do they represent the *same* shape? **Prove your answer!**



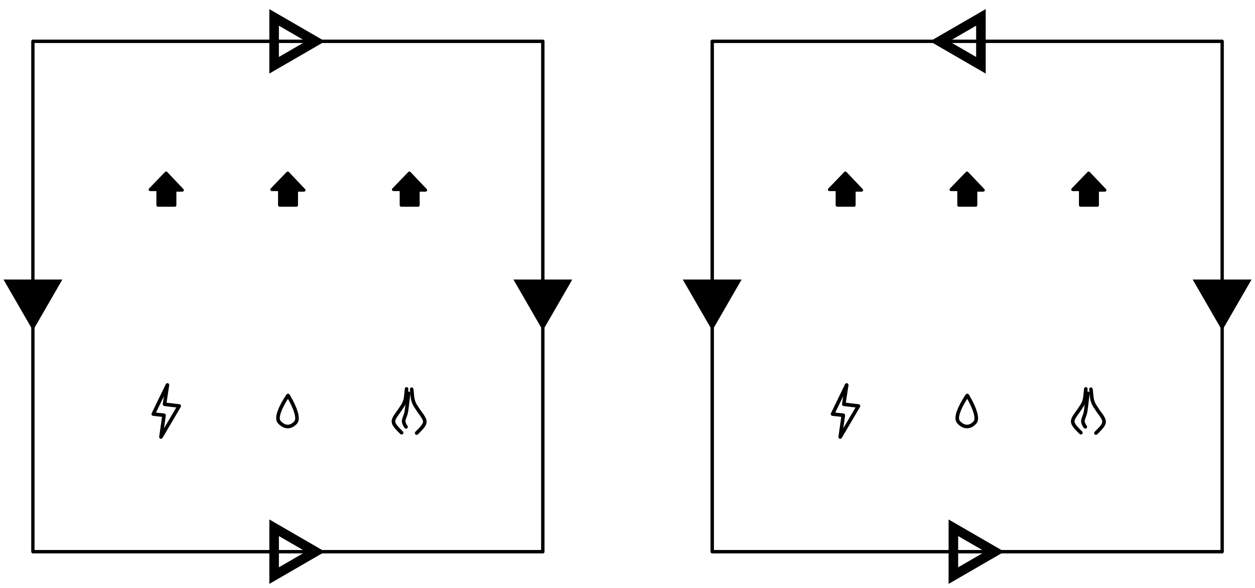




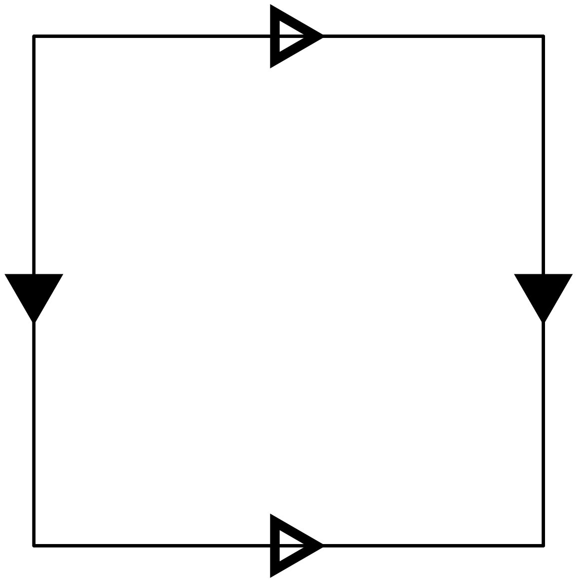
Utility problems

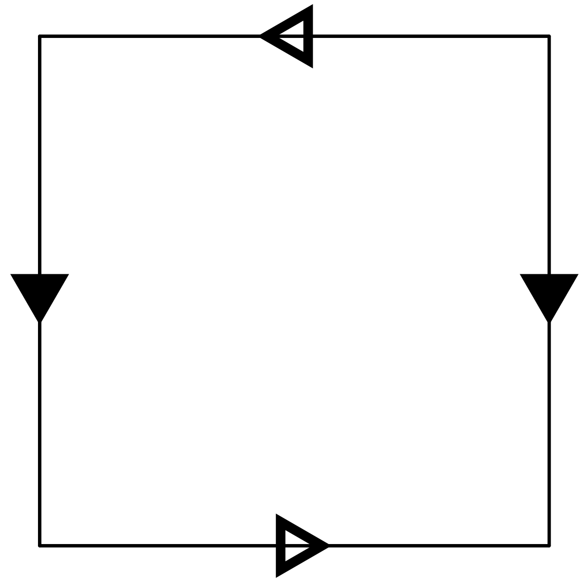
7. Recall the utility problem: *three houses and three utility stations (water, gas, and electricity) are located at different points. Can you connect every house to every utility without the utility lines crossing?* In the plane, the answer was **no**. But on the torus or Klein bottle, the answer is **yes**.

   Draw utility lines on the fundamental polygons below!



8. Let's try a harder one. Now there are *four* houses, and you need to connect them to all three utilities (water, gas, and electricity) without the lines crossing. Can you do this on the torus? On the Klein bottle?





9. You solved it? Well played! But now there are *four* houses and *four* utilities (water, gas, electricity, and lava). You need to connect every house to every utility without the lines crossing. Can you do this on the torus? On the Klein bottle?