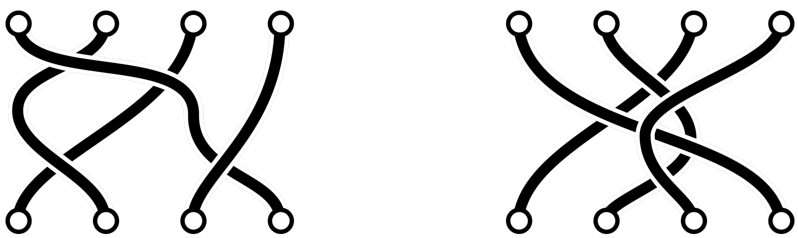
UW Math Circle
Week 16

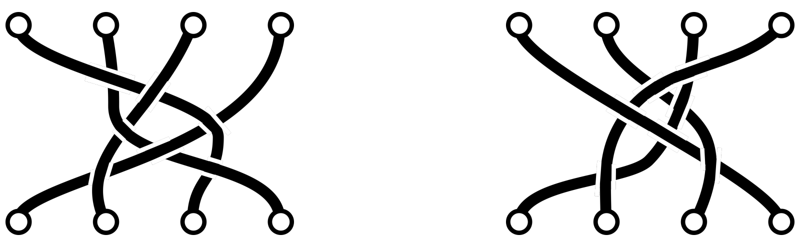
Braids

1. Construct each pair of braids. Then, without unfastening the ends, manipulate the strands to determine whether they are the same or different braids. (If they are different, how can you be sure?)

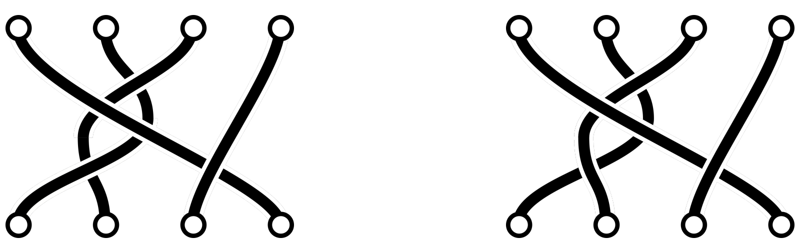
   (a)



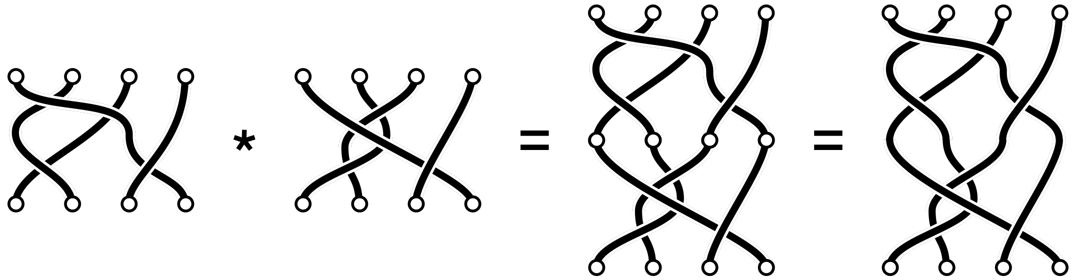
   (b)



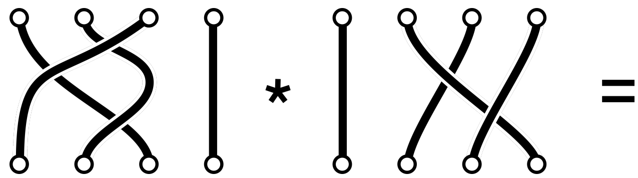
   (c)

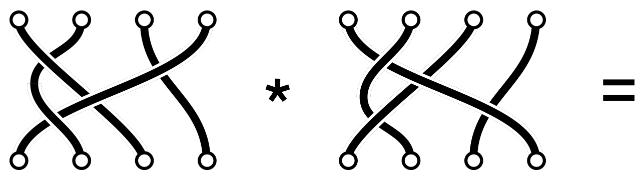


There is an operation that allows us to combine two braids into one new braid! We **compose** two braids together by attaching the ends of the first braid to the beginning of the second:

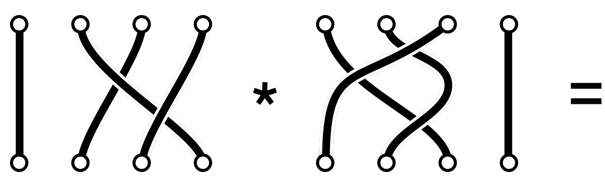


2. Compose the following braids, and draw the new braid that you get:

   (a) 

   (b) 

3. Now compose the braids from part a) in the reverse order, and compare the results.



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Properties of Composition

4. In math, we say two things **commute** if no matter which order we combine them, the result is the same. For example, $1 + 2$ is the same as $2 + 1$ as well.

   Do braids always commute?

5. Find two braids using four strands which *do* commute. (i.e. find two braids where composing them in either order gives the same braid.)

6. If our braids only have two strands, what do our braids look like? Will braids using two strands commute with each other?

Associativity

Sometimes we may want to compose three or more braids. Just like with numbers, we will use parentheses to indicate the order that we compose our braids. **We do the operations inside the parentheses first.**

7. Fill in the following instructions for the possible ways to compose braids A, B, and C:

   (a) To build the braid $(A \star B) \star C$:

   First we connect the *top* of the braid ______ to the *bottom* of the braid ______.

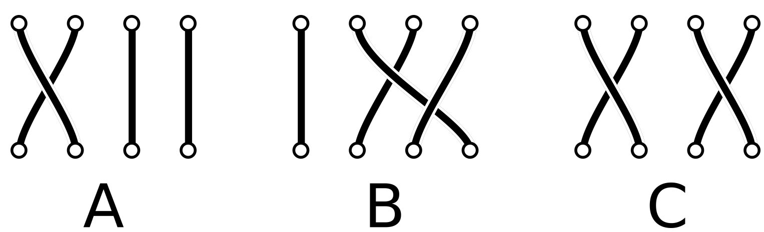
   Then we attach the *bottom* of this braid to the *top* of the braid ______.

   (b) To build the braid $A \star (B \star C)$:

   First we connect the *top* of the braid ______ to the *bottom* of the braid ______.

   Then we attach the *top* of this braid to the *bottom* of the braid ______.

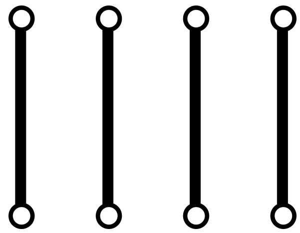
8. Follow these instructions to build the braid $A \star (B \star C)$ and the braid $(A \star B) \star C$ with the braids below. How do the resulting braids compare?



9. Will this be true for every choice of three braids? Why or why not? An operation that satisfies this property is called **associative**.

Inverses and Identity

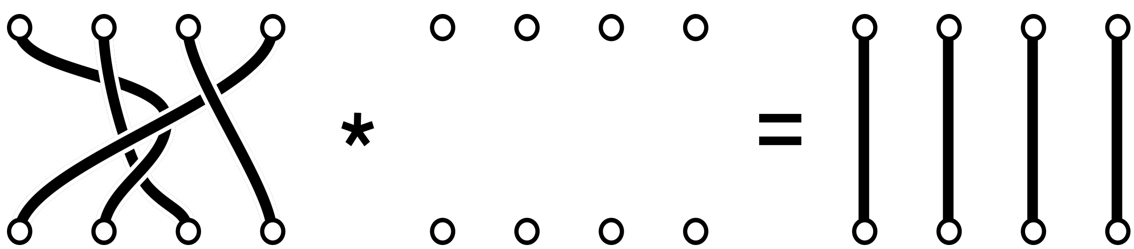
10. The simplest braid is the one with no twists:



What happens when we compose this with another braid? An element with this property is called the **identity**.

11. Notice that in the second example in problem 2, the two braids compose to the identity. (If you didn't notice this before, **go back and check!**)

Find a braid that composes with this one to give the identity.



12. In the previous problem, the braid you found is called the **inverse** of the braid on the left. Does every braid have an inverse?

13. If $B \star A$ is the identity, will $A \star B$ also be the identity?

14. Is it possible for one braid to have two different inverses?

15. Challenge: Other than the identity, is any braid its own inverse?

Groups

A **group** is a set of objects G with an operation $*$ such that

1. $*$ is associative
2. G has an identity
3. Every element in G has an inverse

The most familiar group we know about is the integers under addition! 0 is the identity, and the inverse of any integer n is its negative: $-n$.

In this section we have proven that the **braids using n strands is a group!** We will call this group $\mathbb{B}_n$.

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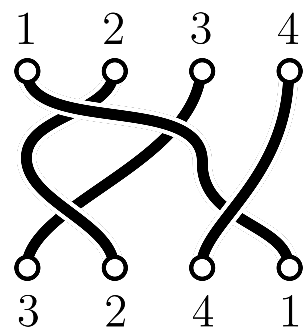
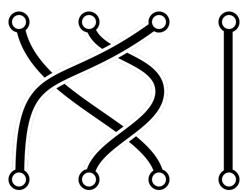
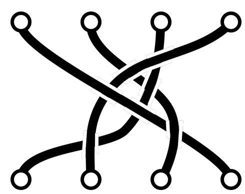
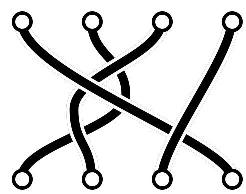
Permutations

Let's keep track of where each strand ends up by labeling them with a number. For example:

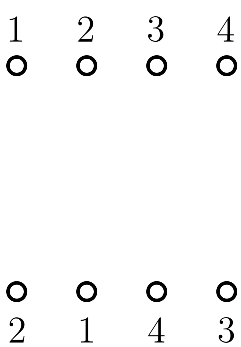


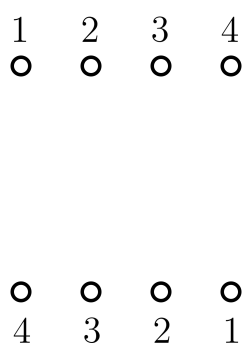
The rearranged numbers we get at the bottom is called a **permutation**. This braid gives the permutation 3241.

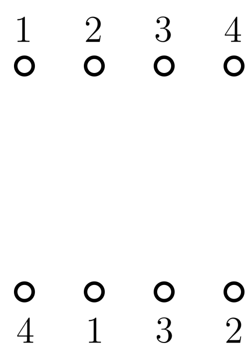
1. What permutations do the following braids give?

2. Draw a braid that results in each of the following permutations:







3. Can two different braids give the same permutation? Give an example or explain why not.

4. Compose the first braid in problem 1 with the second one. What is the resulting permutation?

5. Choose a braid A with permutation 3214 and B with permutation 1243.
   (a) If we compose $A \star B$, what is the resulting permutation?
   (b) Will the answer depend on the braids you choose?

Stop here. Request the next page from your instructor when your group is done.

Permutations as a Group

On the last page we saw that each braid gives us a permutation, and that composition of braids results in a consistent way to compose permutations.

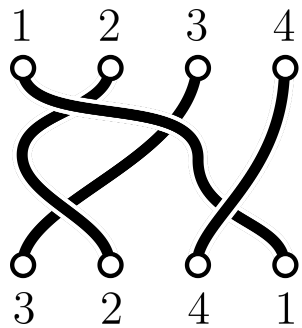
6. Because braids form a group, we will see that permutations do too! However, there are way more braids than permutations.
   (a) How many different permutations of three numbers are there?

   (b) How many different braids that use three strands are there?

7. What is the identity of the permutation group? [Hint: Thank about the identity braid.]

8. Explain why the fact that braids are associative means that permutations will also be associative.

9. Find the inverse of the following braid:



Based on this new braid, what is the inverse of the permutation 3241?

10. Find inverses of the permutations you found braids for in problem 2:

2 1 4 3   4 3 2 1   4 1 3 2

11. Are there any permutations which are their own inverse? Give an example or explain why not.

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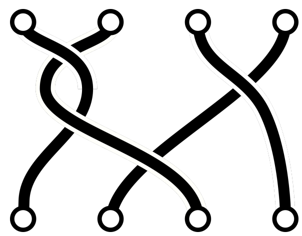
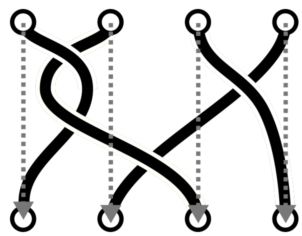
Links

1. (a) Build the following braid:



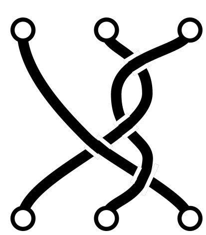
   (b) Take the tops of each strand and connect them to the bottoms of the strands in their new positions:



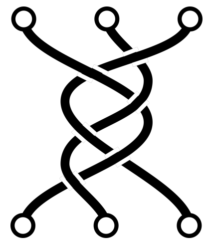
   Careful not to untwist the strands while you do this!

   (c) What does the result look like? There should be two loops connected to each other. What is the length of each loop? [Here we count a single strand as length 1.]

2. Do the same for this braid on three strands. What is the result? How many continuous loops are there?



3. This construction is called a **link**. Draw the link created by this braid:



4. When will a braid result in a link that contains a loop of length 1?

5. Draw a braid on 5 strands that will have two loops, one of length 3 and one of length 2.

6. Can you come up with a general rule for the number and lengths of loops created by any braid?

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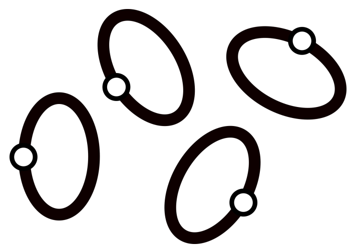
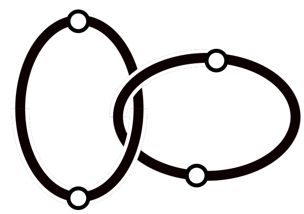
7. Find a braid on four strands for each of the following links:





8. Find two different braids which form the same link.

9. How does a link formed from a braid compare to the link formed from its inverse?

10. Can you construct a braid on 4 strands which makes the same knot as in problem 2? How about on 2 strands? (The length of the loops will be different but make the knot otherwise the same.)

Stop here. Request the next page from your instructor when your group is done.

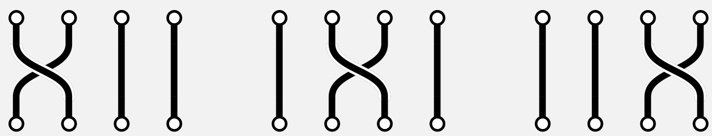
Generators

Every braid can be created by repeating single twists of consecutive strands. We will name the braid which twists the ith strand over the strand to its right b_i, and their inverses b_i^{-1}.

Because every braid on n strands can be created by composing some combination of $b_1, \ldots, b_n$ and their inverses $b_1^{-1}, \ldots, b_n^{-1}$, we say that the braids $b_1, \ldots, b_{n-1}$ **generate** $\mathbb{B}_n$.

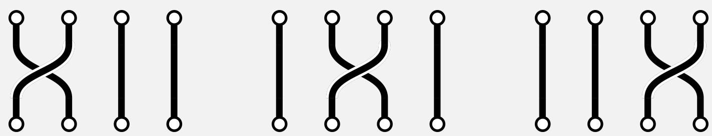
Basic Twists in B_4

Every braid on four strands is some combination of these four twists:



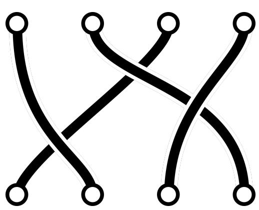
b_1 b_2 b_3

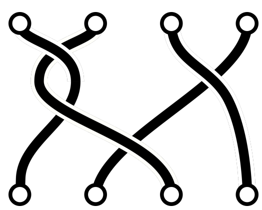
and their inverses:

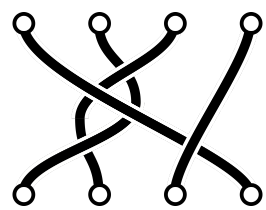


b_1^{-1} b_2^{-1} b_3^{-1}

1. Write these braids as a product of these basic twists.







2. Create the following braids:

$b_1 b_2^{-1} b_1 b_2^{-1}$ $b_1 b_2 b_3 b_2$

3. Show that the braids b_1 and b_3 commute.

4. Show that $b_1 b_2 b_1 = b_2 b_1 b_2$. This is called the **braid relation**.

5. Let D be the braid $b_1 b_2 b_1$.
   (a) Prove that $b_1 D = D b_2$ and $b_2 D = D b_1$.

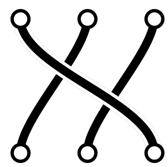
   (b) Use this to prove that D^2 commutes with every braid in $\mathbb{B}_3$.

   [Hint: Since every braid is made from b_1, b_2, and their inverses, you only need to check D commutes with these four braids.]

Stop here. Request the next page from your instructor when your group is done.

The basic twists are an intuitive set of generators for the braid group, but they are not the only ones! **Any set of braids which can be combined, along with their inverses, to create all possible braids generates $\mathbb{B}_n$.**

6. Let t be the braid on three strands given by



   (a) Show that $tb_1t^{-1} = b_2$.

   (b) Explain why this means that $\mathbb{B}_3$ is also generated by b_1 and t.

7. Prove that $\mathbb{B}_3$ cannot be generated by less than two braids. (Hint: Show that if it were generated by one braid, it would have to be commutative!)

8. Originally, we had three generators for $\mathbb{B}_4$. Find *two* braids that generate $\mathbb{B}_4$.

9. What is the least number of braids that can be used to generate all of $\mathbb{B}_n$ for any n?