UW Math Circle

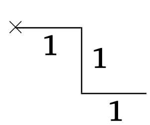
Week 14

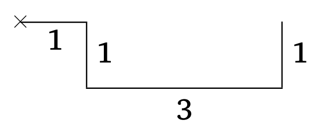
This week we are studying linkages, i.e. mechanisms of rigid, fixed length links (imagine a metal rod) joined at the ends by joints that can rotate freely. Our linkages will all live in the plane, and so are called *planar linkages*. Planar linkages were the subject of much study during the Industrial Revolution, when they were a new and innovative technology.

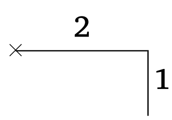
1 Chains

A *chain* is a linkage comprised of a sequence of links joined end to end in a line with one endpoint pinned in place. Figure shows a chain of length 4, with the pinned point marked by an “x”.

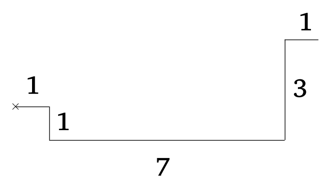
1. Call the region of the plane where the right endpoint can be the *reachable region* of the chain. The reachable region of a chain with a single link is a circle. Draw the reachable regions of the following chains

(a) 

(c) 

(b) 

(cont.)

(d) 

For two regions A, B in the plane, the *Minkowski sum* $A + B$, is the region you obtain by sweeping the center of B along the boundary of A and adding this to A. fig. 1 shows two examples of Minkowski sums.

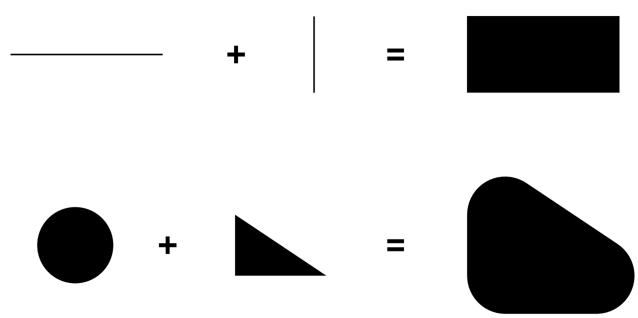
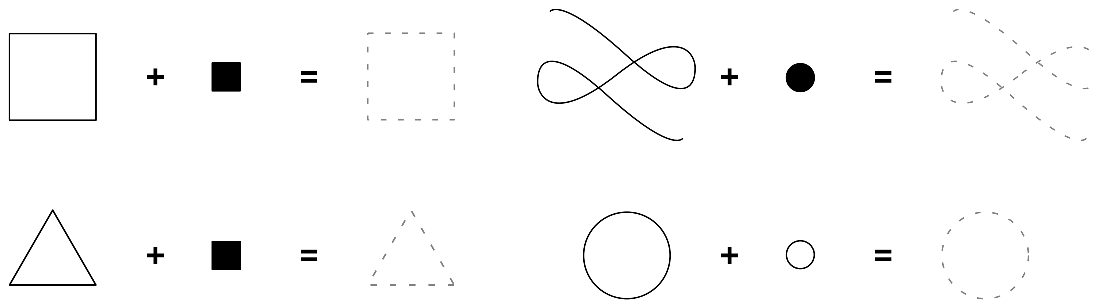


Figure 1: The Minkowski sum of two lines, and the Minkowski sum of a circle with a triangle

2. Draw the following Minkowski sums.



3. The reachable region of a chain with link lengths $r_1, \ldots r_n$ is the Minkowski sum of the circles of radii $r_1, \ldots, r_n$. Use this to check your answers to question 1.

4. A *disk* is a region of points of magnitude at most some constant r, written $\{p : |p| \leq r\}$. (read this as "all the points p such that the magnitude of p is less than or equal to r"). An *annulus* (pl. annuli) is a region of points of magnitude between two constants r_1, r_2, written $\{p : r_1 \leq |p| \leq r_2\}$ ("all the points p such that the magnitude of p is between r_1 and r_2"). Reachable regions of chains are always disks or annuli. Which chains from item 1 have reachable regions that are disks and which have reachable regions that are annuli? What are the respective constants?

   (a)

   (b)

   (c)

   (d)

5. If a chain has link lengths $r_1, \ldots, r_n$, when is the reachable region a disk, and when is it an annulus?

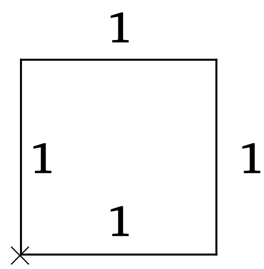
6. How does the reachable region of a chain change if you swap the order of the links? (Hint: Try starting with a two link chain.)

Stop here. Request the next page from your instructor when your group is done.

2 Degrees of Freedom

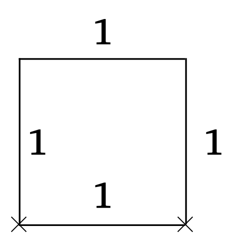
We can make more interesting linkages by joining links in more complicated ways. A useful tool is *degrees of freedom*, which measures the number of angles in a linkage that can move independently.

7. The linkage below has two degrees of freedom. Which angles can move independently? Can you see why three angles cannot move independently?



When we attach a link of length r to a join, that restricts the point's movement; it must always be distance exactly r from the other end of the link, so the linkage loses one degree of freedom. If a linkage has j joins (unpinned) and ℓ links, then the expected number of degrees of freedom for the linkage is $2j - r$.

8. If we pin another join in the above linkage we get the diagram below. This linkage has exactly one degree of freedom; the angle at the bottom left join can change and the rest of the linkage is determined by that angle. Draw the reachable regions for the unpinned joins.



9. Compute the expected degrees of freedom of the above linkage. Why doesn't this agree with the actual number of degrees of freedom?

10. Say a rod between two pinned joins is *redundant*. Can you design a linkage with an even number of rods, none of which are redundant, and with one expected degree of freedom?

11. Design a linkage with negative expected degrees of freedom.

12. Consider a linkage which has joins arranged in an n by m grid, with the first two joins in the bottom row pinned, and links between North-South-East-West neighboring joins in the grid (except between the two pinned joins).

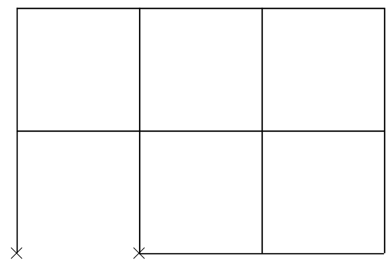
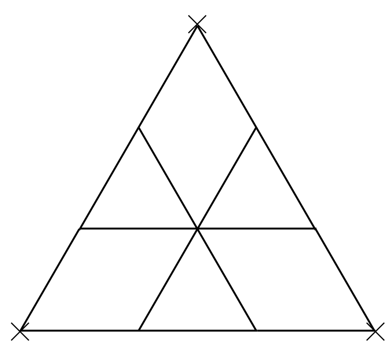


Figure 2: 4×3 grid linkage

(a) How many degrees of freedom does such a linkage have?

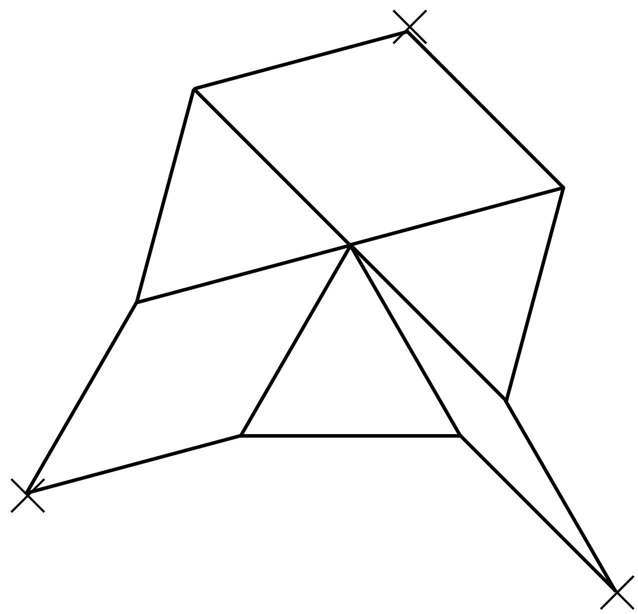
(b) If you add as many links as expected degrees of freedom, the expected number of degrees of freedom will be zero. Can you place exactly that many additional links so that the linkage becomes rigid?

13. (a) How many expected degrees of freedom does the following linkage (which has seven free joins) have?



(b) Is this linkage in fact rigid?

(c) This linkage has the same underlying graph, so the same expected number of degrees of freedom. Is it also rigid?



As we have seen, a linkage can have 0 or negative expected degrees of freedom but still be able to move. Such a linkage is called *overconstrained*.

14. (Challenge) Design your own overconstrained linkage.

Stop here. Request the next page from your instructor when your group is done.

3 Cognate Linkages

Consider fig. 3. Observe that both linkages have one degree of freedom (check this!) and that the indicated points both trace out a circle as the linkages move. These linkages are called *cognates* of one another.

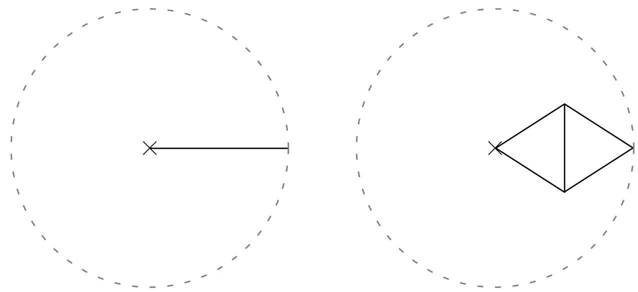
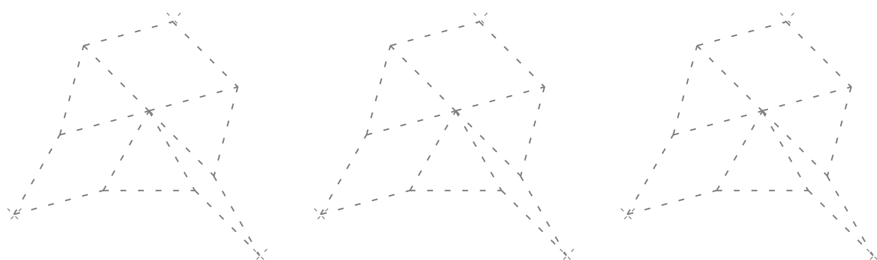


Figure 3

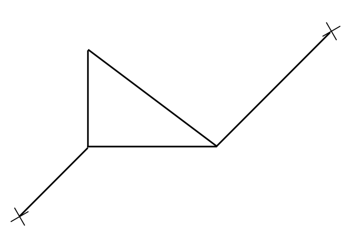
15. Consider the linkage from item 13c.

    (a) Decompose it into three cognate linkages.



    (b) What are the expected degrees of freedom of each of your three linkages?

16. Can you find cognate linkages of the following linkage?



In fact, any linkage involving a triangle connected by links to two pinned points—confusingly called “four-bar linkages”— has two cognate linkages comprised of a triangle connected by links to two pinned points. This is known as the *Roberts-Chebyshev Theorem* discovered independently by British mathematician Samuel Roberts in 1875 and Russian mathematician Pafnuty Chebyshev in 1878.

17. If you have an overconstrained linkage with -5 expected degrees of freedom, how many cognate linkages would you expect to decompose it into?

18. (Challenge) Prove the Roberts-Chebyshev theorem.

Stop here. Request the next page from your instructor when your group is done.

4 Inversion

Define *inversion across the unit circle* (sometimes just *inversion*) to be the transformation taking a point p to the point q of reciprocal magnitude (i.e. $|p| \cdot |q| = 1$) that lies on the ray from the origin to p; figure shows an example. This is defined on the plane except at the origin.

Inversion has the following special property:

(a) Circles centered at the origin go to circles centered at the origin

(b) Lines through the origin go to lines through the origin

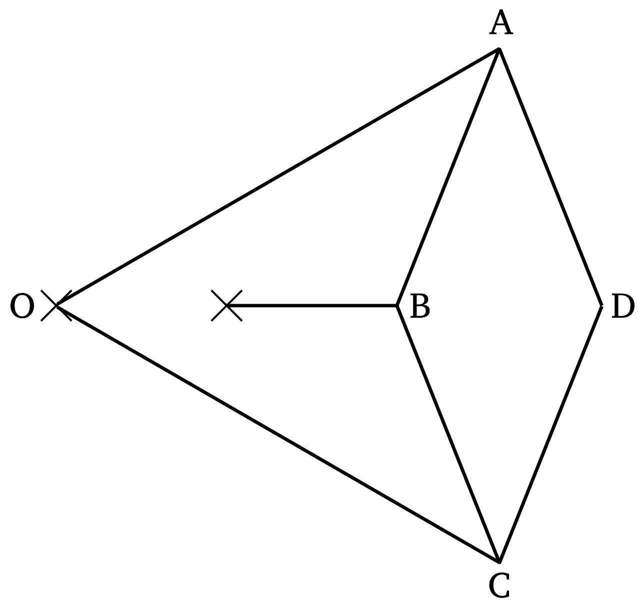
(c) Circles not through the origin go to circles not through the origin.

(d) Circles through the origin go to lines not through the origin

19. Where does the circle centered at the origin of radius 2 go under inversion across the unit circle?

20. Where does the line $y = x$ go under inversion?

21. Consider the linkage below.



The product of distances $\overline{OB} \cdot \overline{OD}$ is always 1. What is the trajectory of D?

22. (Challenge) Show that inversion has the special properties described above.

23. (Challenge) Show that $\overline{OB} \cdot \overline{OD} = 1$.