UW Math Circle

Week 25 – Probability

In **probability**, we study how likely an event is to occur. Examples of events: having rain on a given day; winning the lottery; getting 'Tail' when tossing a coin. For an event A, consisting of possible outcomes of an experiment, we denote the probability that it happens by:

$$P(A) = \frac{\text{Number of outcomes where } A \text{ occurs}}{\text{Total number of possible equally likely outcomes}}.$$

Some examples:

- A standard deck of cards has four suits of cards – clubs (♣), diamonds (♢), hearts (♡), and spades (♠). Each suit has an ace (`A`), the numbers `2`, `3`, `4`, `5`, `6`, `7`, `8`, `9`, and ten (`T`), a jack (`J`), a queen (`Q`), and a king (`K`), with 13 cards in each suit. (*Sometimes decks of cards come with jokers – we ignore these here.*) When drawing a random card from this deck, the probability of drawing a spade is $P(\spadesuit) = \frac{13}{52} = \frac{1}{4}$, and the probability of drawing a ten is $P(\texttt{T}) = \frac{4}{52} = \frac{1}{13}$.
- Coins have a heads (H) side and a tails (T) side. For a fair coin, each side is equally likely to appear, so $P(T) = \frac{1}{2}$ and $P(H) = \frac{1}{2}$.
- A standard die has six sides, with the numbers 1-6 on each side and equally likely to appear. Here, the probability that an even number is rolled is $P(\text{even number}) = \frac{2}{6} = \frac{1}{3}$.

Try these examples:

1. You draw a card from a shuffled standard deck of cards. What is the probability that this card is either a '`6`' or its suit is a spade?

2. Draw two cards from a shuffled standard deck of cards. What is the probability that they are of the same suit? *(Hint: draw them one after the other, without replacing the first into the deck.)*

3. A fair coins is tossed coin three times. What is the probability that the coin comes up tails every time?

4. Abby and Bryan each roll a fair six-sided die. What is the probability that the number that Abby rolled is greater than the number that Bryan did?

Stop here. Request the next page from your instructor when your group is done.

Coins

Let's investigate flipping a fair coin multiple times. First, let's look at flipping a coin four times:

1. What is the probability of flipping the sequence $HTHT$?

2. Fill in the table below for the probabilities of seeing various numbers of heads in four flips:

$P(0 \text{ heads})$	$P(1 \text{ heads})$	$P(2 \text{ heads})$	$P(3 \text{ heads})$	$P(4 \text{ heads})$

3. In general, if I flip a fair coin n times in a row, what is the probability that I get 0 heads? What about n heads?

4. If I flip a fair coin n times in a row, what is the probability that I see exactly 1 head? What about the probability I see at least 1 head?

5. Let's generalize more. What's the probability I see exactly k heads in n flips of a fair coin? (*Hint: It might help to remember that the number of ways to pick a collection of k elements from n is "n choose k", $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.*)

6. Based on your previous answers, what's the most likely number of heads that I could get in n flips of a fair coin? Explain why this is/isn't surprising to you.

Stop here. Request the next page from your instructor when your group is done.

Coins II

Now, instead of letting our number of flips be fixed, let's keep flipping a fair coin until we see a sequence of flips that we want. This is a little hard to reason about one sequence at a time, but we can try to compare which sequences of a given length we are more likely to see in a string of flips.

7. As a warmup, what's the probability that in a sequence of flips that we see a head before a tail?

Abby and Bryan are betting on which of their favorite flip sequences of length 2 will appear first. Let's see who is more likely to see theirs first!

8. Bryan loves heads and really wants to see the sequence HH. Abby likes the sequence HT. What's the probability that Abby sees her sequence before Bryan does?

   (For example: in the sequence of flips $HHTT$, we see the subsequence HH at the very beginning before we see the sequence HT, so Bryan sees his sequence before Abby sees hers. On the other hand, in the sequence of flips $HTHH$, we see the subsequence HT before we see the sequence HH, so Abby sees her sequence before Bryan sees his.)

9. What if Abby preferred the sequence TH instead? Does the probability from the previous problem change?

10. Out of the sequences HH, HT, TH, and TT, which one(s) are more likely to appear before the others in a random sequence of flips?

Now Abby and Bryan want to bet on sequences of length 3. Bryan doesn't like heads *that* much – out of all of the sequences of length 3, he prefers HHT.

11. If Abby's favorite sequence of length 3 is HTH, what is the probability that she sees her favorite sequence in a random sequence of flips before Bryan does?

12. What if Abby's favorite sequence of flips is THH? How does this compare to the likelihood of seeing the sequence HTH before HHT?

13. Out of the eight possible sequences of three flips, which one(s) are the most likely to appear before the others in a sequence of flips? (Are there any sequence(s) that are the most likely?)

Stop here. Request the next page from your instructor when your group is done.

Cards

In various kinds of poker, players are concerned with the probabilities of seeing various patterns when drawing hands of cards.

For what follows, to make our numbers nicer, let's start by temporarily working a deck of cards with just the number cards – in other words, just the ace through ten (`A`, `2`, …, `9`, `T`) of the four suits (♣, ♢, ♡, ♠).

1. What's the probability that when drawing a pair of two cards from this deck of 40 that we get a pair? (In other words, two cards of the same value but different suits?)

2. What's the probability that when we draw a pair of three cards from this deck of 40 that we get a triple (three of the same numbered card, but all different suits?)

3. A poker hand usually consists of five cards. What is the probability that we get exactly a pair in a hand of five randomly chosen cards from this smaller deck?

   (*For example, the hand* `A`♡, `A`♢, `2`♠, `3`♣, `5`♡ *would be a pair, but the hands* `T`♡, `T`♢, `T`♠, `6`♣, `9`♡ *and* `6`♡, `6`♢, `7`♠, `7`♣, `8`♡ *would not be.*)

4. Similarly, what's the probability that a random of hand of five cards would be a triple?

   (*Note that the hand* `T`♡, `T`♢, `T`♠, `6`♣, `9`♡ *would be a triple, but* `7`♡, `7`♢, `7`♠, `8`♣, `8`♡ *would not be a triple.*)

5. If we are looking at a random hand of five cards, it's also possible to hand two pairs in our hand (for example, like `6`♡, `6`♢, `7`♠, `7`♣, `8`♡). What is the probability that we get exactly two pairs in a hand of five randomly chosen cards from this smaller deck?

   (*Note that the hand* `7`♡, `7`♢, `7`♠, `8`♣, `8`♡ *would not be a two pair.*)

6. What is the relative likelihood of the triple, two pair, and pair poker hands?

Stop here. Request the next page from your instructor when your group is done.

Cards II

The next rarest hands in the ranking of poker hands are the *flush* hands and *straight* hands. Flushes are collections of five cards that are all the same suit (for example, A♡, 3♡, 4♡, T♡, Q♡). Straights are collections of five cards whose values are all consecutive (for example, 6♡, 7♢, 8♠, 9♣, T♡, or even 9♡, T♢, J♠, Q♣, K♡ in a standard deck).

To start, for simplicity, let's assume we start with an even smaller deck of cards, using just the ace through eight (A, 2, …, 7, 8) and only three suits (♣, ♢, ♡).

7. With this even smaller deck to draw from, what is the probability that one gets a flush with a randomly-drawn 5-card hand?

8. Similarly, what is the probability that one gets a straight with a randomly-drawn 5-card hand from this even smaller deck? Is this more or less likely than getting a flush?

9. Recalculate the probability of getting a flush and a straight in a 5-card hand drawn from a standard deck of cards. Which is more likely in a randomly-drawn 5-card hand?

10. **(Bonus.)** Let's generalize to decks of cards with s suits and n cards in each suit. Can you find values of (n, s) such that:

- the two pair is more likely than the pair?
- the triple is more likely than the pair?
- the two pair is more likely than the triple?
- the flush is more likely than the straight?
- the straight is more likely than the flush?

Stop here. Request the next page from your instructor when your group is done.

Dice

In the 5th edition of Dungeons and Dragons, players roll a number of different kinds of fair dice, but with varying numbers of sides and sometimes with various ways of deciding the outcome of a dice roll.

1. The most-often rolled die in the game is the d20, with 20 sides numbered 1-20, and each face equally likely. Players are generally interested in rolling a 15 or higher. As a warm-up, what is the probability that a players rolls a 15 or higher on a d20?

2. Players sometimes can be granted *advantage*, meaning that they roll two d20 dice, and take the **higher** of the two as their roll. What is the probability that a player rolls a perfect 20 with advantage? How does this compare to the probability of rolling a 20 on a single d20?

3. What is the probability that a player rolls a 15 or higher with advantage? How does this compare to the probability on one die calculated earlier?

4. Similarly, a player can be granted *disadvantage*, meaning that they roll two d20 dice, and take the **lower** of the two as their roll. What is the probability that a player rolls a perfect 20 with disadvantage? What about rolling a 15 or higher?

5. Some classes can experience *triple advantage*, where they roll 3 d20 dice, and take the maximum value rolled. How much more likely are rolling a perfect 20 and rolling a 15 or higher in this case?

6. In a certain Dungeons and Dragons show, a player was famously granted *quadruple disadvantage*, where they had to roll four d20 dice and take the lowest value rolled. What is the probability of rolling a 15 or higher in this case? What if the player had been granted *quadruple advantage* instead?

Stop here. Request the next page from your instructor when your group is done.

Dice II

7. Suppose we had three dice named X, Y, and Z where die X is more likely to roll a higher number than die Y, and die Y is more likely to roll a higher number than die Z. Should die X be more or less likely to roll a higher number than die Z? Why?

Let's play with some non-standard (but still fair) six-sided dice now. Consider dice A, B, and C, with faces:

- A: 4, 4, 4, 4, 4, 9
- B: 2, 2, 2, 7, 7, 7
- C: 0, 5, 5, 5, 5, 5

8. What are the probabilities that:
   - die A rolls a higher number than die B?
   - die B rolls a higher number than die C?
   - die A rolls a higher number than die C?

9. Now imagine we have two of each die, and we roll them together and take the sum of the numbers rolled. What are the probabilities that:
   - two A dice roll a higher number than two B dice?
   - two B dice roll a higher number than two C dice?
   - two A dice roll a higher number than two C dice?

10. Now let's add in two more dice, D and E:
    - D: 3, 3, 3, 3, 8, 8
    - E: 1, 1, 6, 6, 6, 6

    How likely are D and E to roll higher numbers than the original three dice A, B, and C?

11. Now, again imagine we have two of each of D and E. Now, how likely are two D and two E dice to roll a higher number than two A, B, or C dice?

12. **(Bonus.)** Can you construct your own set of three dice that have the same curious properties as above?

Stop here. Request the next page from your instructor when your group is done.

Miscellaneous

1. (The Monty Hall Problem) You are on a game show where you are asked to pick one of three closed doors. Behind two of the three doors there are goats. But behind one of them, there is a brand new car.

   (a) What is the probability of winning the car?

   (b) You have now picked a door. The game show host (who knows where the car is) opens one of the doors you did not pick which has a goat behind it. Now there are two closed doors and one open door with a goat. The host gives you one last chance to change your door. Should you change your mind and pick the other door? Why or why not?

   (Hint: Does your probability of winning the car change when the host opens one of the doors? The answer to this question is the key to this problem.)

   (c) Now imagine the same game show, but with ten closed doors, where again only one has a brand new car behind it and the other nine have goats. The game show host again takes you through a similar process – pick a door first, and then opens eight of the nine doors that you didn't choose to reveal eight goats. Should you still change your mind and pick the other door? Does this make more intuitive sense?

2. Abby, Bryan, and Charlie are playing in a paintball match. When someone is hit with a paintball, they are out of the game. The three of them are the last three players standing in the match, and have agreed to decide on a winner by trying to hit each other one at a time.

   - Bryan is not an amazing shot – he can only hit his target $\frac{2}{3}$ of the time.
   - Abby is fairly accurate, as she hits her target $\frac{5}{6}$ of the time.
   - Charlie is the best shot out of all three – they can hit their targets $\frac{11}{12}$ of the time.

   The players will get to try to hit each other in order of their accuracy – Bryan first, then Abby, then Charlie. Assume all three players are playing optimally and will try and hit whatever will maximize their chances of winning. What is the probability that Bryan is the last player standing?

Stop here. Request the next page from your instructor when your group is done.