UW Math Circle

Voting Theory

Abby, Bryan, and Charlie are running for president of UW Math Circle. Everyone has cast their ballot with the three candidates listed in order of preference. For example, a ballot reading "$A > B > C$" means the voter prefers Abby over Bryan and Bryan over Charlie.

Look inside your ballot packet for the ballot slips! How do we know who won?

1. In a **Plurality** voting system, the candidate with the most first-choice votes wins the election. Which candidate wins under this voting system?

2. In a **Instant-runoff/"Ranked choice"** voting system, we eliminate the candidate with the fewest first-choice votes. We cross that candidate off everyone's ballot (so if we eliminate candidate C, then "$A > C > B$" becomes "$A > B$"). Then we start over with the updated ballots, always eliminating the candidate with the fewest first-choice votes. The last candidate remaining wins the election. Which candidate wins now?

3. In a **Round-robin** voting system, We match each pair of candidates head-to-head. The winner of each matchup is the candidate ranked higher on most ballots (so Charlie beats Abby in their matchup, because 5/9 ballots ranked Charlie above Abby). The candidate who wins the most matchups wins the election. Which candidate wins now?

Josh and Sean enter the election late, and everyone updates their ballot to include the new candidates. Everyone keeps the original candidates in the same order—so if a ballot originally said "$A > B > C$", now it might say "$S > A > B > J > C$", but it won't put B above A or C above B.

The math circle supervisor says, "If neither Josh nor Sean wins the election, the outcome shouldn't change. For example, if Abby would have won the original election, then the new winner should be Josh, Sean, or still Abby—otherwise our voting system is no good!" A voting system that always satisfies this condition has **independence of irrelevant alternatives**.

4. Does each voting system from the previous page have independence of irrelevant alternatives? Why or why not?

   (a) **Plurality**

   (b) **Instant-runoff/"Ranked Choice"**

   (c) **Round-robin**

Stop here. Request the next page from your instructor when your group is done.

Voting system criteria

Here are some more conditions we might like our voting system to have.

- *Majority criterion (MAJ)*: If candidate X is ranked first on more than half of the ballots, candidate X should win the election.
- *Condorcet criterion (COND)*: If candidate X would beat every other candidate in a head-to-head matchup, candidate X should win the election.
- *Cool criterion (COOL)*: If candidate X loses a head-to-head matchup against at least one other candidate, candidate X should not win the election.
- *Consistency criterion (CON)*: Suppose we hold the election twice with two different groups of voters, and candidate X wins both times. If we combine all the ballots into one vote, candidate X should still win.

5. Fill out ≥ 8 of the blanks in the following table.

   If the voting system fails a criterion, write **F** and give an example where it fails.

   If the voting system satisfies a criterion, write **S** and explain why!

	MAJ	*COND*	*COOL*	*CON*
Plurality				
Instant-runoff				
Round-Robin				

6. (Bonus) Can you design a voting system that satisfies each of the conditions above? Here are some additional voting systems to consider:

   - **Borda count**: The canidate ranked last on a ballot receives one point, the candidate ranked second-to-last receives two points, and so on. The candidate with the most total points wins the election.
   - **Approval**: Instead of ranking candidates, a voter marks each candidate "approve" or "disapprove". The candidate with the most "approve" votes wins the election.
   - **Random ballot**: One voter's ballot is chosen as random. The candidate ranked first on this ballot wins.

Stop here. Request the next page from your instructor when your group is done.

Design a nightmare election

7. Design a "nightmare election": that is, an election between three or more candidates and a list of ballots where as many voting systems as possible produce a different outcome.

Plurality winner:

Instant-runoff/"Ranked Choice" winner:

Round-Robin winner:

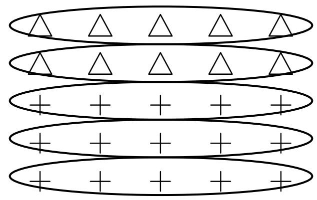
Borda count winner:

Approval winner:

Random ballot winner:

Gerrymandering

Suppose there are 50 students in math circle. 20 always want to do geometry (△) and 30 always want to do algebra (+). They sit in class according to the following grid:



They are put into 5 (contiguously sitting) groups of 5. If there are more geometers than algebraists in a group they will do geometry and vice versa. When grouped as above, 2 groups will do geometry and 3 groups will do algebra.

8. Can you make 5 contiguous equal size groups, so that every group does algebra?

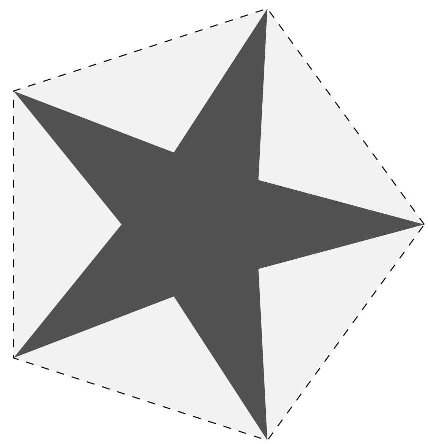
△	△	△	△	△
△	△	△	△	△
+	+	+	+	+
+	+	+	+	+
+	+	+	+	+

9. Can you make 5 contigous equal size groups, so that 3 groups do geometry?

△	△	△	△	△
△	△	△	△	△
+	+	+	+	+
+	+	+	+	+
+	+	+	+	+

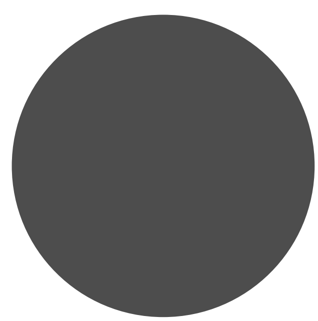
In politics, the redrawing of electoral district boundaries to give your candidate or party the advantage is called **gerrymandering**.

The *convex hull* of a shape is everything inside the new shape formed by stretching a rubber band around the original.



The convex hull of this star is the entire filled in pentagon.

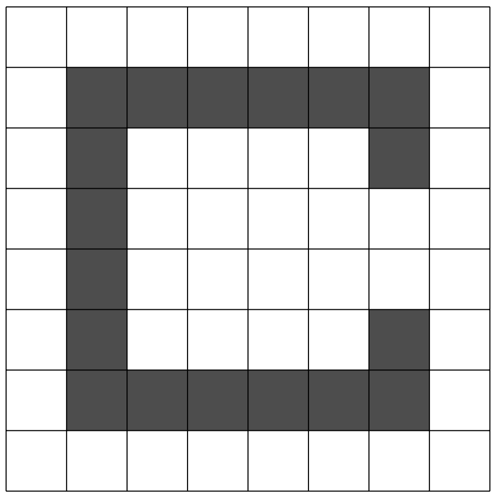
10. Shade in the convex hull for the following shapes.



One way to detect gerrymandering is to to compute the **Convex Hull score** of the district's shape. The Convex Hull score of a district D is given by:

$$\text{CH score of D} = \frac{\text{Area of D}}{\text{Area of the convex hull of D}}$$

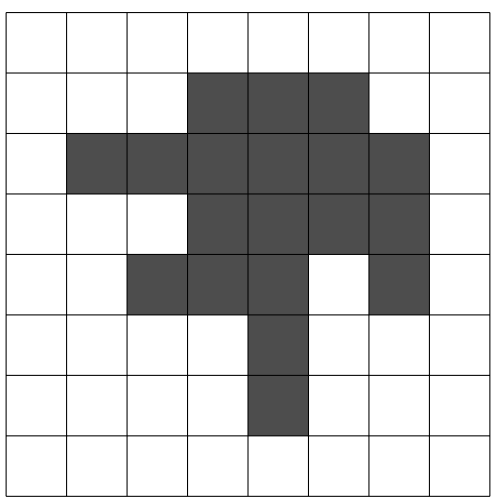
11. Compute the CH score of the following shaded districts.



Area of district =

Area of convex hull =

CH score =



Area of district =

Area of convex hull =

CH score =

12. Notice that $0 \leq (\text{CH score of D}) \leq 1$. Which do you think is a sign of gerrymandering:

- CH score closer to 1
- CH score clower to 0?

Why?

13. Now there are 60 students in the math circle and 26 want to do geometry (△) and 34 want to do algebra (+). Once again, if there are more geometers than algebraists in a group then they will do geometry and vice versa.

    Can you make 4 contiguous equal size groups of cells, so that 3 groups do geometry and each groups shape has CH score greater than .6? How about .65?

△	△	+	+	+	+	△	+	△	△
+	+	+	△	△	△	△	+	+	△
+	△	△	+	+	+	+	△	△	+
+	+	△	+	+	△	△	+	+	△
△	△	△	△	+	+	△	+	+	△
+	+	+	+	+	△	+	+	+	△

△	△	+	+	+	+	△	+	△	△
+	+	+	△	△	△	△	+	+	△
+	△	△	+	+	+	+	△	△	+
+	+	△	+	+	△	△	+	+	△
△	△	△	△	+	+	△	+	+	△
+	+	+	+	+	△	+	+	+	△