UW Math Circle

Week 22 – Combinatorics

Tips:

- If n is a natural number, then $n!$ or "n factorial" is the product of all integers from 1 to n. For example,

  $$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

  If you have to arrange n different objects in a line, there are $n!$ possible ways to order them.

- If n and m are natural numbers with $n \geq m$, then $\binom{n}{m}$ or "n choose m" is the number of ways to pick m objects from a set of n objects. To calculate it:

  $$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{n \times (n-1) \times \cdots \times (n-m+2) \times (n-m+1)}{m \times (m-1) \times \cdots \times 2 \times 1}.$$

  For example, let's say there are ten Math Circle instructors, and we need a team of three instructors to write a worksheet. The number of possible teams we could pick is

  $$\binom{10}{3} = \frac{10!}{3!(10-3)!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120.$$

1. You have seven shirts, three pairs of pants, and two pairs of shoes in your closet. You need to put on an outfit consisting of a shirt, a pair of pants, and a pair of shoes. How many different outfits are possible?

2. Five swimmers compete in a race. In how many different orders can the swimmers finish the race?

3. Septimus Heap has five textbooks: Magyk, Flyte, Physik, Fyre, and Darke Magyk.

   (a) In how many ways can Septimus arrange his textbooks on his shelf?

   (b) Septimus decides to study three subjects today. In how many ways can he choose three of these textbooks to study?

4. How many diagonals does a regular hexagon have? A regular octagon? How about a regular n-gon?

Stop here. Request the next page from your instructor when your group is done.

5. A certain cell has six mitochondria, colored red, orange, yellow, green, blue, and purple. The cell wants to undergo mitosis and split into two cells, with three mitochondria in each. How many ways can the cell split up its mitochondria? For example, one way would be to put the red, orange, and yellow mitochondria together and the green, blue, and purple mitochondria together. Another way would be to put the red, yellow, and blue together and the orange, green, and purple together.

6. There are 10 math circle instructors: Abby, Bryan, Charlie, Clare, Glenn, Jenny, Joe, Josh, other Josh, and Sean. We need a team of three instructors to write a worksheet. However, Sean and Clare cannot work together, because analysts and combinatorialists don't get along. Furthermore, the two Josh's cannot work together because they would just goof off and make Josh puns. Given this, how many ways are there to form a team of three?

7. A theater troupe has ten members with ages 19, 21, 24, 25, 29, 34, 35, 48, 50, and 53. How many ways are there to pick a producer, a director, and a stage manager for their next production, assuming the producer must be older than the director and the director must be older than the stage manager?

8. You have 4 black beads and 3 white beads and a piece of string.
   (a) In how many different ways can you thread these beads onto a string?
   (b) After threading the beads, you tie the ends of the string together to make a necklace. How many different ways could you arrange the beads on the necklace (we now consider two arrangements of beads the same if you can rotate one arrangement to make the other).

9. Ada has decided that the word PERMUTATION is boring, so she wants to coin a new word for it. However, to make sure the new term catches on, she's decided to use exactly the same letters—that is, her new word must be an anagram of PERMUTATION. How many options does Ada have for her new word?

10. If a taxicab can only drive north and east, in how many ways can the cab drive 5 blocks north and 4 blocks east?

Stop here. Request the next page from your instructor when your group is done.

11. (a) You have five hot dogs and you want to put condiments on them. You have ketchup and mustard, and you want to put condiments on all five hot dogs, but you don't like mixing condiments, so you put either ketchup or mustard on each dog. How many ways can you do this?

    (b) You realize you also like plain hot dogs, so now you want to put ketchup or mustard on some of your hot dogs but you don't have to put condiments on all of them. How many ways can you do this?

    (c) You decide you actually *don't* like plain hot dogs, and you want to try mixing condiments, so now all five hot dogs must have at least one condiment, and you can put both ketchup and mustard on your hot dogs. How many ways can you do this?

Tip:

- Imagine you have n stars and you want to put them into groups $1, 2, \ldots, k$, where the groups are allowed to be empty. Line up the stars and choose $k-1$ places between stars to put $k-1$ bars, then say group 1 is all the stars to the left of the first bar, group 2 is all the stars to the left of the second bar, and so on, finally saying group k is all the stars that are to the *right* of the last bar. For example, in the arrangement below, group 1 has two stars, group 2 has three stars, group 3 has no stars, and group 4 has one star.

$$\star\star \mid \star\star\star \mid\mid \star$$

  What we are really doing is starting with $n+k-1$ spaces, picking any $k-1$ of them to put bars in, and filling the remaining spaces with stars. We can therefore calculate the number of ways to do this as $\binom{n+k-1}{k-1}$.
  Use this "stars and bars" strategy to solve the problems in this section.

12. (a) You have a bunch of rubber balls and two buckets: red, and yellow. How many ways are there to place rubber balls into buckets so that the total number of balls in buckets is at most 7?

    (b) Now add an orange bucket. How many ways are there to place rubber balls into the three buckets so that the total number of balls in buckets is exactly 7?

13. A *monomial* in the variables x, y, z is a product of the form $x^a y^b z^c$, and we call the sum of the exponents, $a + b + c$, the *degree* of this monomial. The exponents can be any natural number $0, 1, 2, \dots$. For example, $x^2y^2z^2$ is a monomial of degree 6, and $x^5y^0z^1$ is also a monomial of degree 6. How many monomials of degree at most d are there in the variables x, y, z?

    HINT: If you have a monomial $x^a y^b z^c$ of degree less than d, you can make it degree d by multiplying by a new variable w with exponent $d - (a + b + c)$. Then $x^a y^b z^c w^{d-(a+b+c)}$ is degree exactly d. How many monomials of degree *exactly* d are there in the four variables x, y, z, w? How does this relate to the original problem?

Stop here. Request the next page from your instructor when your group is done.

14. We place the numbers 1-9 in a three by three grid. The grid is called a *semimagic square* if each row and each column adds to the same total. How many different semimagic squares are there? Rotations and reflections of a semimagic square count as *different* semimagic squares.

15. How many ways can you arrange n pairs of opening and closing parentheses, "(" and ")", so that the opening and closing pairs "match up" in the usual way?

    These "match up": (())(), ((())), ()()()

    These do NOT "match up": ())((),)()()(

n	1	2	3	4	5	6
# of ways						

16. How many ways can you divide n students into subgroups?

    Given students A,B,C,D,E some possible subgroups are:

    (AC)(BDE), (A)(BE)(CD), (ACDE)(B).

n	1	2	3	4	5	6
# of ways						