# UW Math Circle

Week 21 – Teamwork Puzzles

## Hat Puzzles

In each puzzle below, your group stands in a circle and an instructor randomly places a **blue** or **green** hat on each student's head so that you can see everyone else's hat but not your own. All students will guess their hat color simultaneously: to guess blue, make the ASL sign for **"B"**, to guess green, make the ASL sign for **"G"**, and if you do not know, **do nothing**. *No other communication is allowed.* Your group wins if *everyone* guesses their hat color correctly. If no one guesses incorrectly, you may guess again (without removing hats) until you win. However, if *anyone* guesses incorrectly, your group loses and you must try again with new hats.

ASL sign for "G"

ASL sign for "B"

1. The instructor gives everyone a green hat except one randomly chosen student who receives a blue hat.

2. An instructor randomly distributes hat colors, but there will be at most 1 blue hat.

3. An instructor randomly distributes hat colors, but there will be at least 1 green hat.

In each puzzle below, your group stands in a straight line facing forward and an instructor will distribute hats. Each student sees the hats of everyone ahead of them, but not their own or the hat of anyone behind them. Starting from the back, each student says either **"green"** or **"blue"** and no other communication allowed. However, you may plan beforehand.

4. Your group wins if at least half guess correctly (for odd group sizes, round down; for example, a group of 5 needs only 2 correct). Find and demonstrate a winning strategy.

5. Only three students line up and your group wins if at least two guess correctly. Find and demonstrate a winning strategy.

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**Stop here.** Request the next page from your instructor when your group is done.

# More Hat Puzzles

Here is one winning strategy for Problem 5.

- First, we give "Blue" the number "0" and we give "Green" the number "1".
- **Back student:** Sum the values of the two hat colors in front of them. Say "blue" if the sum is even and "green" if the sum is odd.
- **Middle student:** Use the back student's guess together with the front student's hat color to deduce your own color.
- **Front student:** Use both previous guesses to deduce your own color.

*Example:* Suppose the sequence is "Blue, Green, Green". Then the back student sees "Green, Green" and sums $1 + 1 = 2$ (even), then says "Blue." The middle student knows the back student's sum was even; since the front hat is green (i.e. "1"), their own hat must also be green (i.e. "1") to keep the sum even, so they say "green." The front student uses similar logic: because the back student's sum was even and the middle student announced their hat color is green (i.e. "1"), the front student deduces their hat color must be green (i.e. "1") to keep the sum even.

6. Line up and perform a solution to Problem 5 using the strategy described above. Explain why this strategy works.

7. Now *everyone* in your group lines up and your group wins if you make at most one incorrect guess. Find and demonstrate a winning strategy.

8. Again, everyone in your group lines up and your group wins if you make at most one incorrect guess, but now the hat colors could be **green**, **blue**, or **red**.

Now, some of your group will stand in a circle. Then, without communicating, every student will simultaneously guess their hat color by making the ASL sign corresponding to the first letter of the color. However, now there is only one round of guessing and your group wins if at least one person guesses their hat color correctly. You may plan beforehand.

ASL sign for "R"

ASL sign for "G"

ASL sign for "B"

9. Break into pairs (so the "circle" only has two students). Your instructor will randomly give each student a **red** hat or a **green** hat. Find and demonstrate a winning strategy.

10. Form a circle with three students. Your instructor will randomly give each student a **red** hat, a **green** hat, or a **blue** hat. Find and demonstrate a winning strategy.

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**Stop here.** Request the next page from your instructor when your group is done.

## Even More Hat Puzzles

11. Form a circle with all students in your group. Each student will have a **green** or **blue** hat, and there will be at most one blue hat. The instructor randomly selects a valid hat configuration, then distributes hats accordingly. There is one simultaneous round of guessing: each student signs **"G"** for green, **"B"** for blue, or **does nothing** to pass. Your group wins if at least one student does not pass and there are no incorrect guesses. Find and demonstrate a strategy that maximizes your chance of winning.

12. Form a circle with three students and the instructor randomly places a **green** or **blue** hat on each student and all students guess simultaneously by either signing **"G"** for green, **"B"** for blue, or doing **nothing** to pass. Your group wins if at least one student does not pass and there are no incorrect guesses. Find and demonstrate a strategy that maximizes your chance of winning.

13. (Challenge) Suppose seven students are in a circle and an instructor randomly distributes a **blue** or **green** hat to each student. Again, there is one simultaneous round of guessing: each student signs **"G"** for green, **"B"** for blue, or **does nothing** to pass. The group wins if at least one student does not pass and there are no incorrect guesses. Find and demonstrate a strategy so that you win with probability 7/8.

*Hint:* If there is one student that observes all blue hats or all green hats so that they know the hat sequence is very close to "BBBBBBB" or "GGGGGGG", then you can use a strategy similar to Problem 11. In general, there will be one student that observes the hat sequence is very close to one of the following:

| | | | |
|---|---|---|---|
| BBBBBBB | GGGBBBG | GGBBBGB | BBGBBGG |
| GBGBGBB | BGBBGBG | BGGBGGB | GBBBGGG |
| BGGGBBB | GBBGBBG | GBGGBGB | BGBGBGG |
| GGBGGBB | BBGGGBG | BBBGGGB | GGGGGGG |

*Note:* These hat sequences are exactly the binary sequences used for Hamming(7,4) code.