UW Math Circle

Week 6 – Recursion

A *recursive* sequence is a sequence of numbers where future terms in the sequence depend on the values of the sequence coming before it. We will play with several such recursive sequences and study their properties.

Fibonacci Numbers

Start with the numbers 0 and 1 as the first two numbers of an infinite sequence of numbers. The *Fibonacci numbers* F_n are generated by making the next number the sum of the previous two numbers. In symbols, we say that $F_n = F_{n-1} + F_{n-2}$ for any $n \geq 2$, giving:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144\ldots$$
$$\text{or}\quad F_0,\ F_1,\ F_2,\ F_3,\ F_4,\ F_5,\ F_6,\ F_7,\ F_8,\ F_9,\ F_{10},\ F_{11},\ F_{12}\ldots$$

$$\begin{aligned} F_2 &= 1 \quad \text{because} \quad F_2 = F_1 + F_0 = 1 + 0 = 1 \\ F_3 &= 2 \quad \text{because} \quad F_3 = F_2 + F_1 = 1 + 1 = 2 \\ F_4 &= 3 \quad \text{because} \quad F_4 = F_3 + F_2 = 2 + 1 = 3 \\ F_5 &= 5 \quad \text{because} \quad F_5 = F_4 + F_3 = 3 + 2 = 5 \end{aligned}$$

We call the equation $F_n = F_{n-1} + F_{n-2}$ the *recursive formula* for the Fibonacci numbers.

1. Fill out the following table tracking the sum of the first n Fibonacci numbers, which we call $S_n = F_1 + F_2 + \ldots F_n$. For example, $S_3 = F_1 + F_2 + F_3 = 1 + 1 + 2 = 4$.

n	1	2	3	4	5	6	7	8	9
S_n			**4**						

   Can you come up with a simpler general formula for S_n in terms of Fibonacci numbers?

2. Now consider the sum of the first n *odd-indexed* Fibonacci numbers, which we call $O_n = F_1 + F_3 + \cdots + F_{2n-1}$. For example, $O_3 = F_1 + F_3 + F_5 = 1 + 2 + 5 = 8$:

n	1	2	3	4	5	6
O_n			**8**			

Similarly, let E_n be the sum of the first n *even-indexed* Fibonacci numbers (so $E_n = F_2 + F_4 + \cdots + F_{2n}$). For example, $E_3 = F_2 + F_4 + F_6 = 1 + 3 + 8 = 12$:

n	1	2	3	4	5	6
E_n			**12**			

Fill out the above tables for O_n and E_n. Again, can you come up with simpler general formulas for O_n and E_n in terms of Fibonacci numbers?

3. Let A_n be the average of the nth Fibonacci number and the $(n+3)$rd Fibonacci number, so $A_n = \frac{1}{2}(F_n + F_{n+3})$. For example, $A_3 = \frac{1}{2}(F_3 + F_6) = \frac{1}{2}(2 + 8) = 5$.

n	1	2	3	4	5	6	7	8	9
A_n			**5**						

Can you come up with a simple general formula for A_n in terms of Fibonacci numbers?

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Tiling with Squares and Rectangles

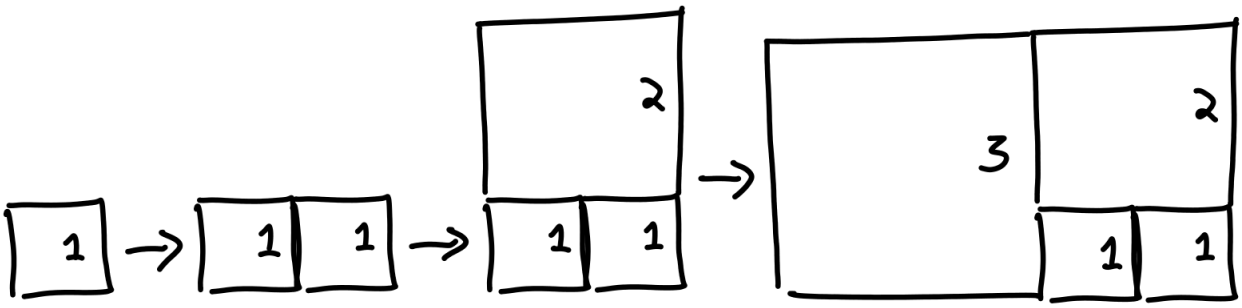
4. Here's how to draw a *golden rectangle* with 4 squares:

   Step 1: Draw a 1×1 square.

   Step 2: Attach a second 1×1 square.

   Step 3: Attach a 2×2 square to the long side of your rectangle from step 2.

   Step 4: Attach a 3×3 square to the long side of your rectangle from step 3.

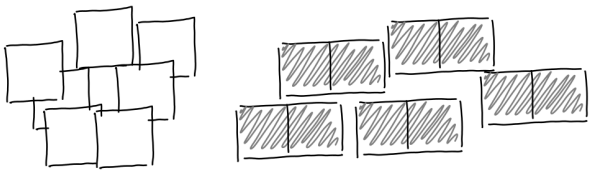


   You can keep going to make a golden rectangle with n squares: at each step, just attach a new square to the long side of your current rectangle. What are the dimensions of a golden rectangle with 10 squares? *(Hint: try to track the dimensions of golden rectangles with each number of squares! Is there a pattern?)*

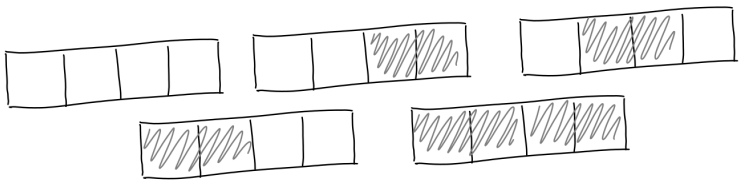
n	1	2	3	4	5	6	7	8	9	10
length										
width										

5. Let $\square_n$ be the sum of the first n *squares* of Fibonacci numbers, so $\square_n = F_1^2 + F_2^2 + \cdots + F_n^2$. Using a golden rectangle with n squares, can you come up with a simple general formula for $\square_n$ in terms of Fibonacci numbers?

6. You have a pile of white 1×1 squares and grey 1×2 dominoes.



Let T_n be the number of ways to fill a $1 \times n$ rectangle with 1×1 squares and 1×2 dominoes. For example, $T_4 = 5$, since you can use these squares and dominoes to fill up a 1×4 rectangle in 5 different ways:



Fill in the following table:

n	1	2	3	4	5	6
T_n				**5**		

Can you come up with a general formula for T_n in terms of Fibonacci numbers?

STOP **Stop here.** Request the next page from your instructor when your group is done.

7. Let's prove your general formula for T_n in terms of Fibonacci numbers holds for all n!

(a) How many ways are there to tile a $1 \times n$ rectangle, assuming that the right-most tile is a square? Try some small cases and make a guess in terms of the number of ways to tile a $1 \times n$ rectangle T_n.

n	1	2	3	4	5	6
$-\square$ tilings						

(b) What about the number of ways to tile a $1 \times n$ rectangle, assuming that the right-most tile is a domino? Again, try some small cases and make a guess in terms of the number of ways to tile a $1 \times n$ rectangle T_n.

n	1	2	3	4	5	6
$-\square\square$ tilings						

(c) Notice that *every* tiling of a $1 \times n$ rectangle by squares and dominoes ends on the right with either a square or a domino. Using this observation, write down a (recursive) formula for T_n.

(d) Let's now perform this proof using a process called *mathematical induction*!

i. Hopefully you've already done this: verify your formula for T_n in terms of Fibonacci numbers works for $n = 1, 2$. (These are called the *base cases*.)

ii. Now, **assume** that your formula for T_n holds for $n = 1, 2, \ldots, k-1, k$ for some k. Use part (c) to conclude that your formula also holds for $n = k + 1$. (This is the critical step of the proof, called the *inductive step*.)

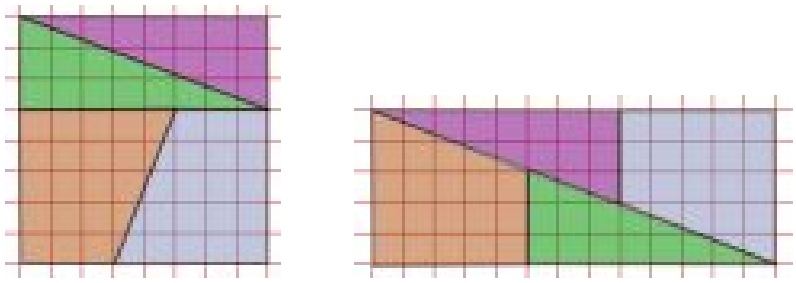
iii. We now claim that we can conclude that your formula for T_n holds for any n we like!

- Use i. and ii. to conclude that your formula holds for T_3.
- Do it again! Use i., ii., and the previous conclusion to conclude that your formula holds for T_4.
- Keep going! What would you have to do to conclude that your formula holds for an arbitrary n?

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Paradox of the Illusory Area

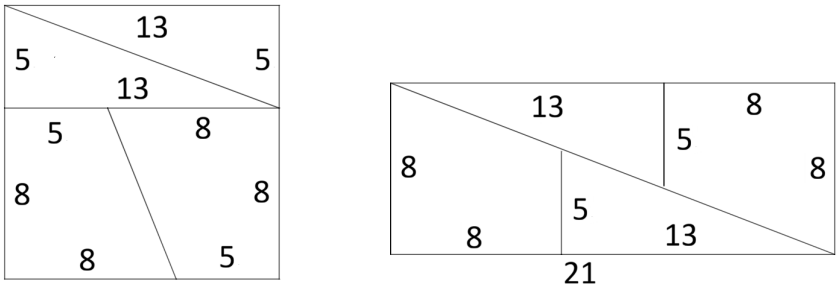
Consider dissecting an 8×8 square into four pieces, as shown, and reassemble them to form a 5×13 rectangle:



The original area of our four pieces was $8 \times 8 = 64$, but we've now used them to cover a rectangle with area $5 \times 13 = 65$. What?!

8. Explain how we have magically "gained" one unit of area. (One possible explanation is $64 = 65$... but obviously this can't be correct!)

9. Similarly, dissect the following 13×13 square, and reassemble them to form a 8×21 as shown:



Now, one unit of area has disappeared from the square to form the rectangle! How?!

10. Consider the sequence given by $I_n = F_{n-1}F_{n+1} - F_n^2$ (where we take $F_0 = 0$). Complete the following table of values of I_n:

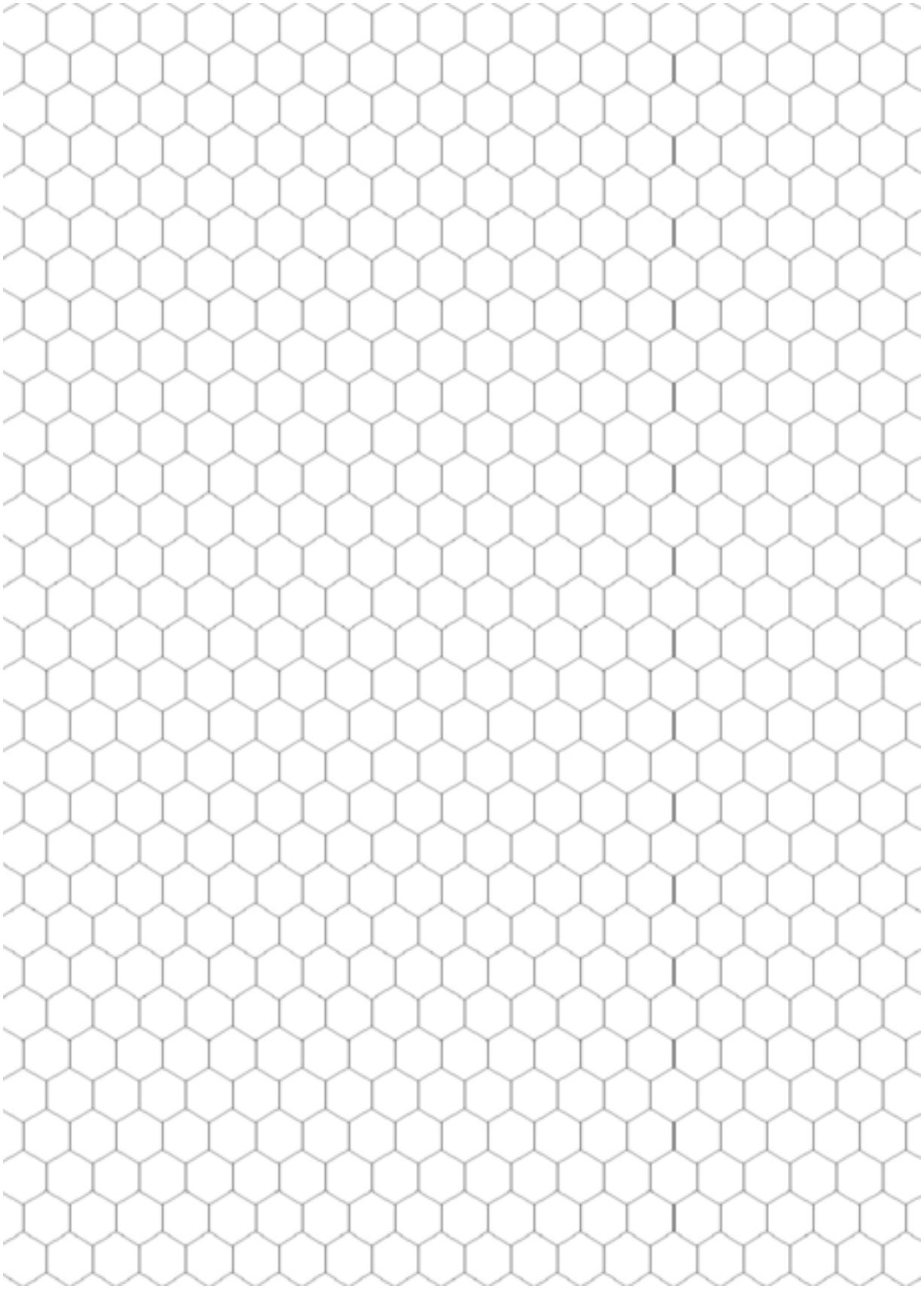
n	1	2	3	4	5	6	7	8
I_n								

Make a guess for what I_n should be!

11. Based on your calculations above, use the sequence I_n to cook up geometric dissections with "magically appearing/disappearing" area. *(Hint: you've already seen $n = 6, 7$.)*

12. (Challenge) Try to prove your pattern for I_n holds!

Stop here. Request the next page from your instructor when your group is done.



Pascal's Triangle

Let's now consider a two-dimensional recursive pattern. Imagine an infinite row of 0's with a single 1:

0 0 0 0 0 0 1 0 0 0 0 0 0

Now, we can generate new rows in the "gaps" of the previous row by adding the two numbers above them together. For example, here are the next three rows of this pattern:

0 0 0 0 0 0 1 0 0 0 0 0 0
0 0 0 0 0 1 1 0 0 0 0 0
0 0 0 0 0 1 2 1 0 0 0 0 0
0 0 0 0 1 3 3 1 0 0 0 0

We call the resulting pattern of non-zero numbers *Pascal's Triangle*.

13. Using the given hexagonal grid paper, create a Pascal's triangle with at least 8 rows (including your first row with a single 1).

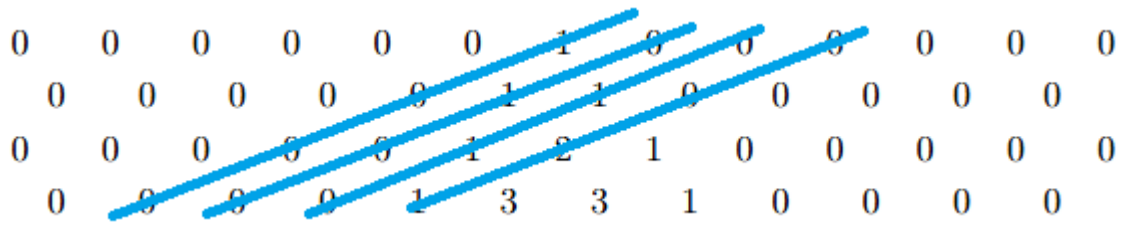
14. Shade in the even-numbered hexagons in your Pascal's Triangle. What patterns do you see?

15. (a) On a copy of your Pascal's Triangle, shade in the hexagons with numbers that are divisible by 3.

    (b) On another copy of your Pascal's Triangle, shade in the hexagons with numbers that are divisible by 5.

    What patterns do you see?

16. Add up the numbers in your Pascal's Triangle along the following "diagonals":



Make a guess for the general pattern of these sums. (Challenge: Try to prove your pattern always holds!)

17. (Challenge) Extending your three copies of Pascal's Triangle to 16 rows or more, continue to shade in the hexagons with numbers that are:

    (a) divisible by 2.

    (b) divisible by 3.

    (c) divisible by 5.

    To save time, you might want to consider all of your numbers mod 2, 3, or 5. (Ask an instructor if you don't remember what that means!) Do your patterns from before continue to hold? (Challenge: Can you prove that they continue to hold for all n?)

Stop here. Request the next page from your instructor when your group is done.

Going the Distance

In general, we can use *mathematical induction* to algebraically prove the patterns you came up with on the first page of the worksheet hold for all n.

18. You guessed that S_n, the sum of the first n Fibonacci numbers from Problem 1, was equal to some $f(n)$ (probably in terms of Fibonacci numbers). Let's prove this by induction!

    (a) Hopefully, your formula holds for $n = 1$. Verify that $S_1 = f(1)$ (the *base case*).

    (b) Now, let's suppose your formula holds for $n = 1, 2, \ldots, k-1, k$, so $S_k = f(k)$. Note that $S_{k+1} = S_k + F_{k+1} = f(k) + F_{k+1}$. Use the recursion for the Fibonacci numbers to show that $S_{k+1} = f(k+1)$ (the *inductive step*).

    (c) Using the same reasoning as from Problem 7(d)iii, we can now conclude that $S_n = f(n)$ for all n. Hooray!

19. Prove that at least one of your guesses for O_n and E_n from before holds for all n. Suppose your formula for the one you pick is $g(n)$.

    (a) Verify that $____1 = g(1)$ (the *base case*).

    (b) Now, let's suppose your formula holds for $n = 1, 2, \ldots, k-1, k$, so $____k = g(k)$. Note that $____{k+1} = ____k + ____ = g(k) + ____$. Use the recursion for the Fibonacci numbers to show that $____{k+1} = g(k+1)$ (the *inductive step*).

    (c) Using the same reasoning as from Problem 7(d)iii, we can now conclude that $____n = g(n)$ for all n. Hooray!

We may also use *double-counting* arguments to prove identities about Fibonacci numbers! By Problem 7, we know the number of ways to tile an $1 \times n$ rectangle with squares and dominoes is $T_n = F_{n+1}$.

20. Let's count the number of ways to tile a $1 \times (n+1)$ rectangle with squares and dominoes that use at least one domino.

    (a) Let's try a small example first. Let $n = 5$ (so $n+1 = 6$). How many ways are there to tile a 1×6 rectangle with 0, 1, 2, 3 dominoes?

dominoes	0	1	2	3
tilings				

    We can also think about what squares the right-most domino in our tiling covers. How many tilings are there where the right-most domino in the tiling covers each of the following pairs of consecutive squares?

right-most domino	(1,2)	(2,3)	(3,4)	(4,5)	(5,6)
tilings					

    (b) In general, how many ways are there to tile a $1 \times (n+1)$ rectangle with zero dominoes? Use this to count the number of ways to tile an $1 \times (n+1)$ rectangle while using at least one domino.

(c) If you have a tiling of an $1 \times (n+1)$ rectangle that uses at least one domino, there must be a right-most domino in the tiling. Suppose the right-most domino covers squares $k+1$ and $k+2$, for some integer k between 0 and $n-1$, inclusive. How many ways are there to tile an $1 \times (n+1)$ rectangle in this scenario?

(d) Use (c) to prove there are $F_1 + F_2 + \cdots + F_n$ ways to tile a $1 \times (n+1)$ rectangle while using at least one domino. What identity does this recover?

21. (Challenge) For any positive integers m and n, show that $F_{m+n} = F_m F_n + F_{m-1} F_{n-1}$. To do this, consider the number of ways to tile a $1 \times (m+n-1)$ rectangle with squares and dominoes. Some questions to ponder about squares m and $m+1$ in your tiling:

- What happens if these two squares are covered by a single domino? How many ways are there to complete the tiling in this case?
- What happens otherwise? How many ways are there to tile the rectangle?

22. (Challenge) Use double-counting to prove the patterns you came up with for the sequences O_n and E_n. Some quantities that you might be inspired to count in an interesting ways:

    - The number of ways to tile a $1 \times (2n-1)$ rectangular board with squares and dominoes, broken up by the number of dominoes used at the end of a tiling.

    - The number of ways to tile a $1 \times (2n)$ rectangular board with squares and dominoes *(using at least one square)*, broken up by the number of dominoes used at the end of a tiling.

Stop here. Request the next page from your instructor when your group is done.