UW Math Circle

Week 5 – Fold and Cut

1 Fold and Cut

The *Fold and Cut Theorem*—first proven in 1999 by Anna Lubiw with father-and-son duo Martin and Erik Demaine—roughly states that any drawing you can make with straight lines can be cut out with a single straight cut if the paper it was drawn on is first folded in an appropriate way. Today we will work out how to do this ourselves!

1. Work out how to fold and cut out an equilateral triangle with a single cut straight across the paper. What is the fewest number of folds you need?

2. Work out how to fold and cut out a square with a single cut. What is the fewest number of folds you need?

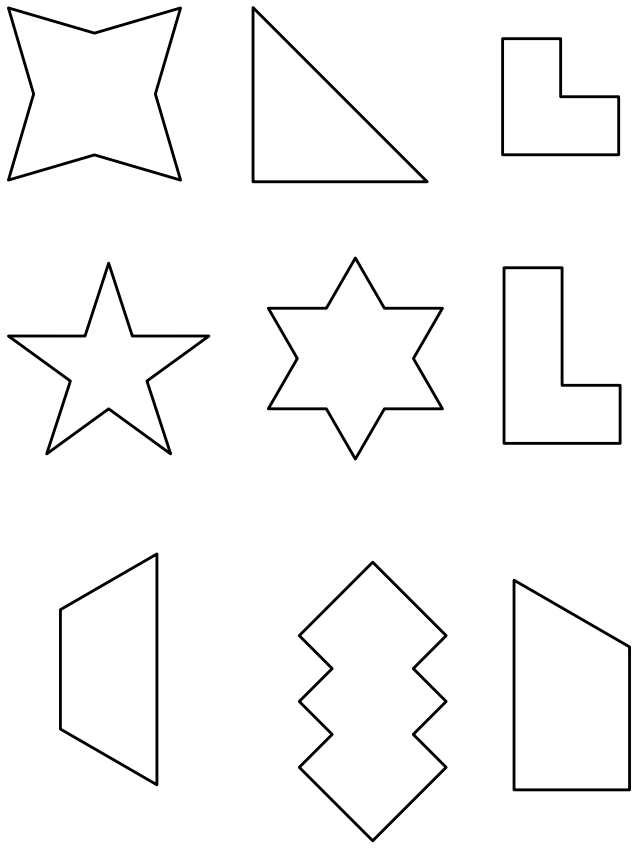
3. Do the same with the regular pentagon, hexagon, heptagon, octagon, nonagon, and decagon. Record how many folds were required for each in the table below.

n-gon	folds
3	
4	
5	
6	
7	
8	
9	

4. What pattern do you see? Can you explain why that pattern holds?

Stop here. Request the next page from your instructor when your group is done.

5. Cut out at least four of the shapes below by folding the paper and making one cut straight across. Show an instructor when you are done



6. _____

Stop here. Request the next page from your instructor when your group is done.

2 Decomposing Shapes

A line going through a vertex of a polygon which evenly divides the angle is called a *angle bisector*. The dotted line below is an angle bisector of a vertex of a triangle. We mark a line as an angle bisector by drawing an arc across the angle which is bisected.

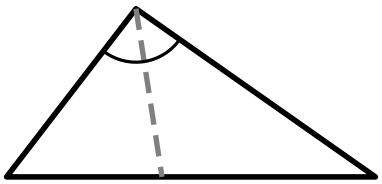
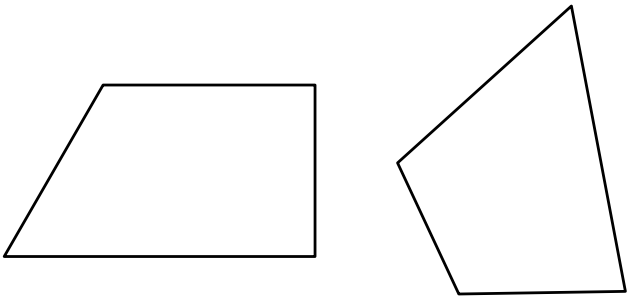


Figure 1: angle bisector

7. If you fold along an angle bisector, what happens to the two edges making up the angle?

8. Draw angle bisectors for each vertex of the polygons below.



A line which makes a 90° angle with another line is said to be *perpendicular* to the second line. We mark two lines as perpendicular with a right angle as seen below. A line perpendicular to an edge of a triangle and passing through the opposite vertex is called an *altitude* of the triangle. The dotted line of the triangle below is an altitude.

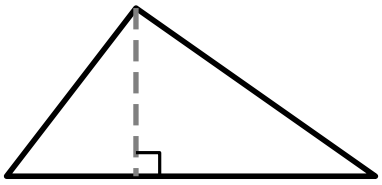
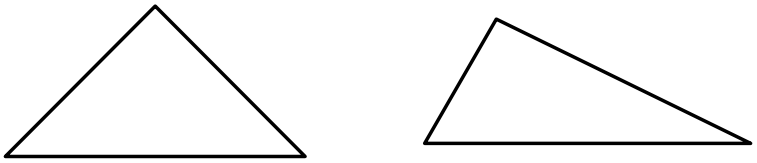


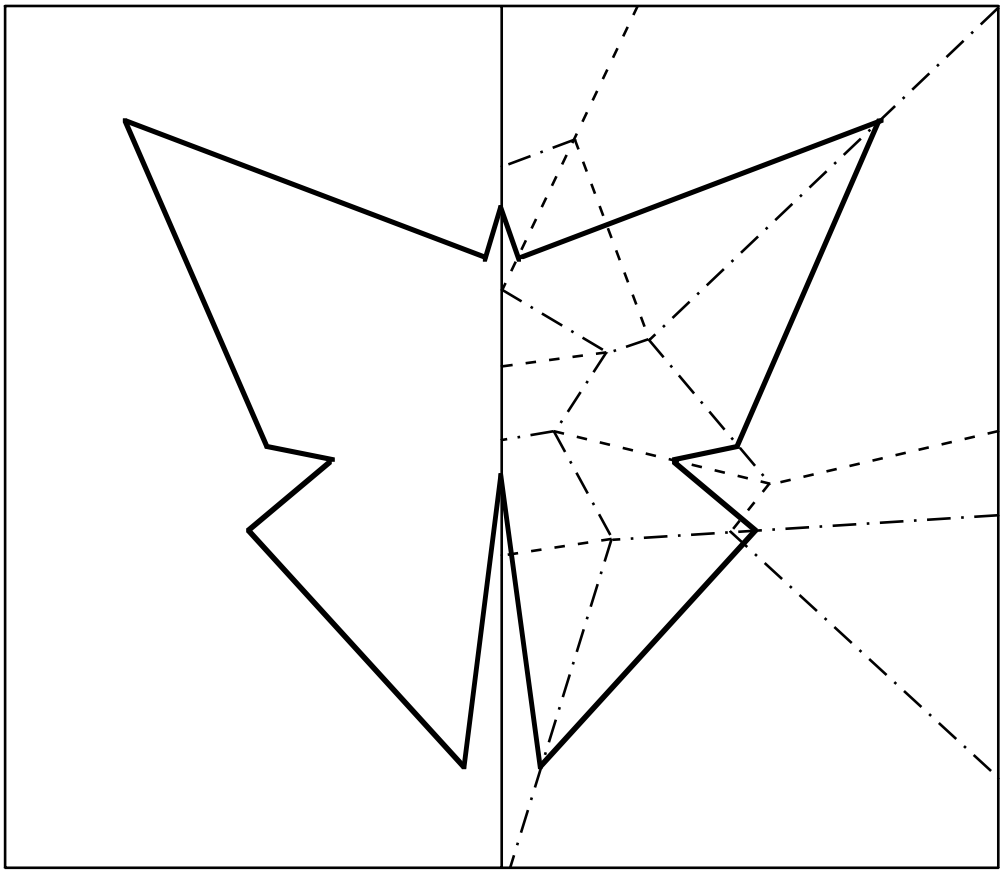
Figure 2: an altitude of a triangle

9. If you fold along a line perpendicular to a second line, what happens to that second line?

10. Draw all three altitudes of each triangle below.



11. Label the angle bisectors and perpendicular lines in the figure below.



Stop here. Request the next page from your instructor when your group is done.

3 The Straight Skeleton Method

Given a figure we want to cut out with one cut, we can construct a way of folding the paper using the figure's *straight skeleton*. We construct the straight skeleton by shrinking the figure inwards, every edge moving along a perpendicular path at an equal speed, as if the edges of our figure caught fire and burned at an even speed everywhere.

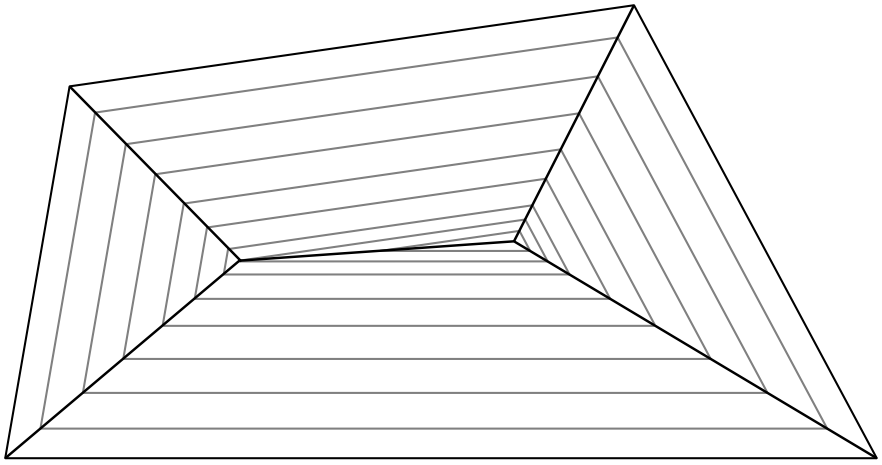


Figure 3: the straight skeleton of a figure formed by tracing vertices as it shrinks

As the figure shrinks, the vertices will follow angle bisectors. When two vertices meet, they will combine into a new vertex, and this new vertex will move along a new angle bisector. Tracing the vertices as this happens, we obtain the straight skeleton.

The straight skeleton of a figure may not give all the folds you need to flatten your paper! You will also need to add lines that 1) are perpendicular to the edges of your figure and 2) pass through vertices of the straight skeleton.

12. Shape 12 is a quadrilateral with its straight skeleton and a perpendicular line as described above drawn on. Fold along these lines and add one more fold passing through a vertex of the straight skeleton to fold the figure flat, and cut it out with one cut.

13. Shape 13 is a non-convex quadrilateral. Cut it out with a single cut!

Stop here. Request the next page from your instructor when your group is done.

14. Ask an instructor for one of the challenge figures to cut out.

15. Design a figure for another student to cut out.

Stop here. Request the next page from your instructor when your group is done.