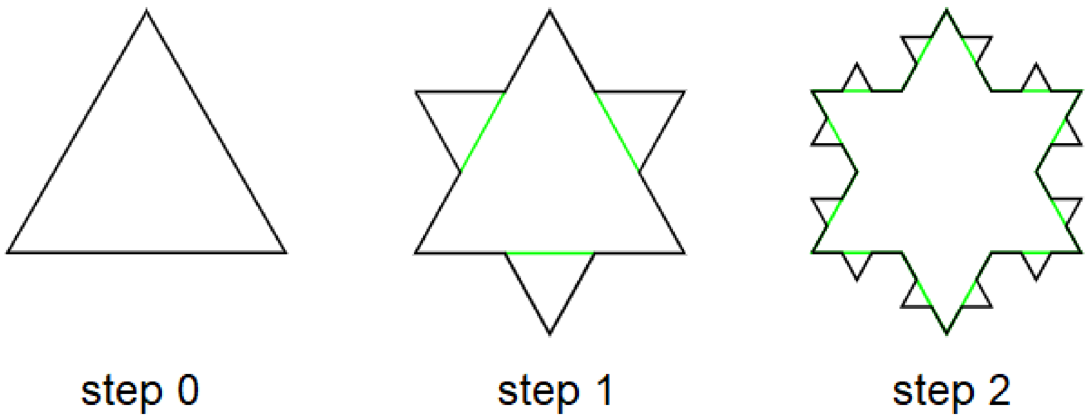
Name: ______________________________

UW Math Circle

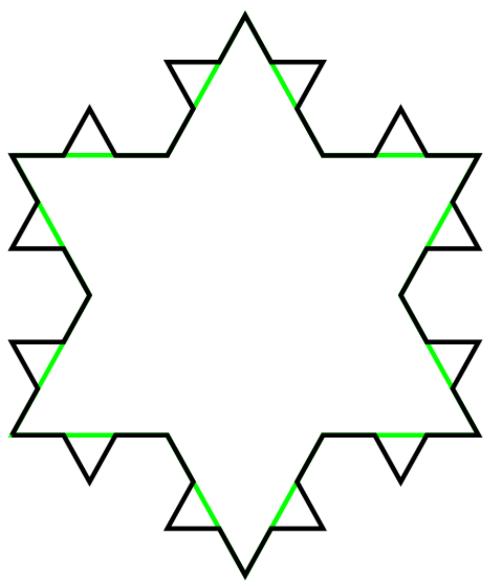
Week 19 - Fractals

1 The Koch Snowflake

Let's draw a snowflake! Start with an equilateral triangle. To draw the next step, cut each line segment of the perimeter into thirds, and attach an equilateral triangle to the middle third.



1. Draw step 3 of the snowflake.



2. Let's call the original side length 1. Since step 0 is an equilateral triangle, it has perimeter 3 and area $\frac{\sqrt{3}}{4}$. What are the perimeter and area of the next three steps? (Note: the area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$.)

step	**perimeter**	**area**
0	3	$\frac{\sqrt{3}}{4}$
1		
2		
3		

3. A *geometric series* is a series of numbers where each term is r times the previous term, for some fixed ratio r. For example,

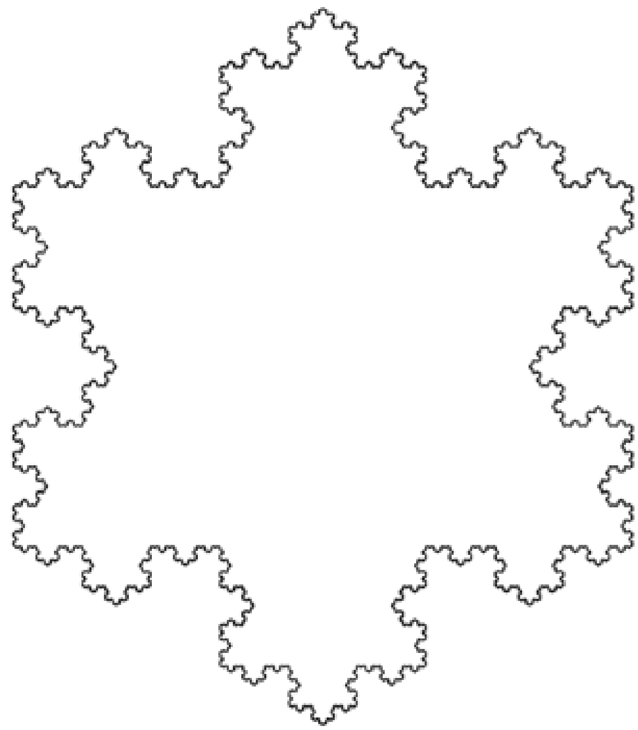
$$10 + 5 + \frac{5}{2} + \frac{5}{4} + \cdots$$

is a geometric series, because each term is $\frac{1}{2}$ times the previous term. For geometric series with positive terms, you can find the sum using this rule:

- If $r < 1$, then $a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \cdots = \frac{a}{1-r}$.
- If $r \geq 1$, then $a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \cdots = \infty$.

What are the sums of these geometric series?

$$10 + 5 + \frac{5}{2} + \frac{5}{4} + \cdots =$$
$$\pi + \frac{2\pi}{3} + \frac{4\pi}{9} + \frac{8\pi}{27} + \cdots =$$
$$1 + \frac{5}{2} + \frac{25}{4} + \frac{125}{8} + \cdots =$$



If you had time to draw infinitely many steps, you'd get a fractal called the *Koch snowflake*!

4. What's the area of the Koch snowflake?

5. What's the perimeter of the Koch snowflake?

2 Fractal Dimension

We can classify shapes by their *dimension*.

- **0-dimensional**: a point.
- **1-dimensional**: shapes with length but no area (eg. a line, a curve).
- **2-dimensional**: flat shapes with area but no volume (eg. a sheet of paper, a polygon).
- **3-dimensional**: solid shapes (eg. a cube, a ball, you).

In this section, we'll see why fractals can have weird dimensions!

6. Logarithms! If x and y are numbers, then $\log_x y$ is **the power you need to raise x to, to get y**. For example, $\log_2 32 = 5$, because you need to raise 2 to the power of 5 to get 32. (In other words, $32 = 2^5$.)

   Evaluate the following logarithms.

$$\begin{aligned} \log_2 64 &= \\ \log_3 27 &= \\ \log_5 25 &= \\ \log_5 5 &= \end{aligned}$$

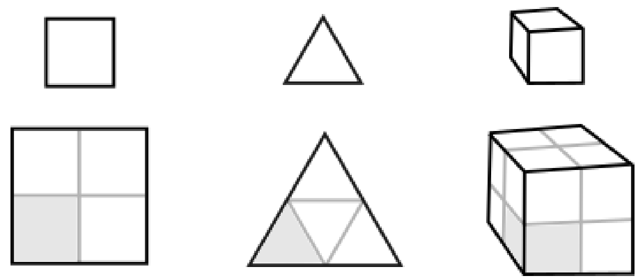
7. Even if x and y are integers, $\log_x y$ can sometimes be a decimal. Without using a calculator, **what numbers should go before the decimal point?**

$$\begin{aligned} \log_2 3 &= \\ \log_3 20 &= \\ \log_2 100 &= \end{aligned}$$

   When you're done, use a calculator to check your answers.

8. Let's *scale* some shapes! Scaling a shape by n means stretching it by a factor of n in every direction, so it becomes n times as long, n times as wide, etc.

   A square, a triangle, and a cube are scaled by 2. The square is now 4 times as big (we can cut it into four copies of the original). Similarly, the triangle is 4 times as big, and the cube is 8 times as big.

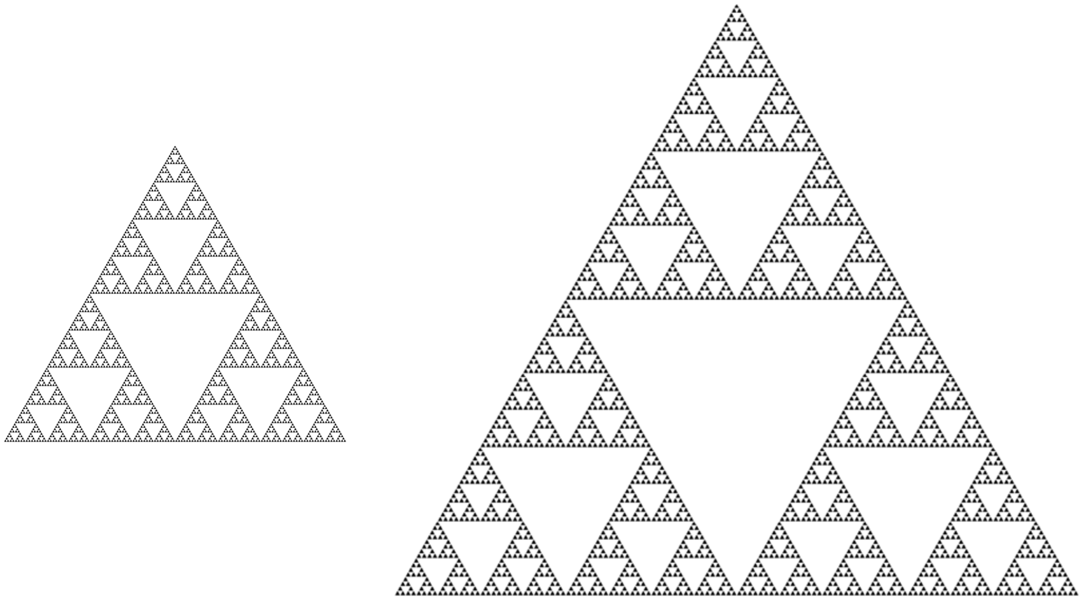


   If you scale these shapes by 3, how many times as big will they get? What if you scale them by 4? **Use your answers to fill in the table.**

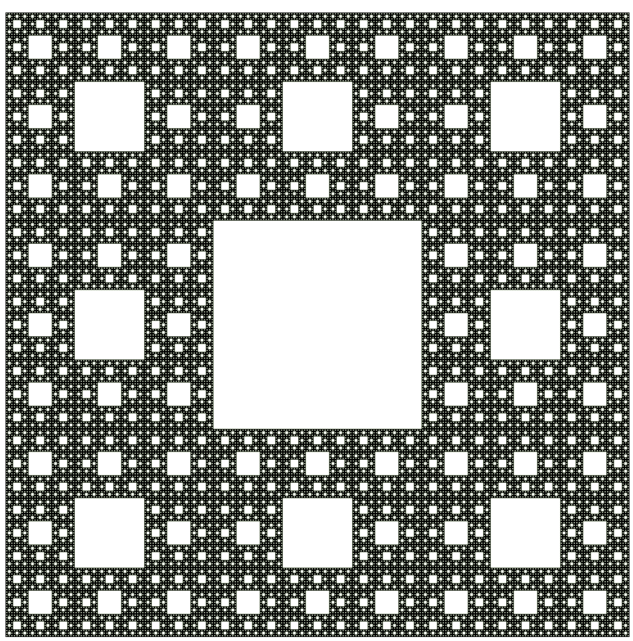
square		
scale	size multiplier	$\log_{\text{scale}}$(size multiplier)
2	4	
3		
4		
triangle		
scale	size multiplier	$\log_{\text{scale}}$(size multiplier)
2	4	
3		
4		
cube		
scale	size multiplier	$\log_{\text{scale}}$(size multiplier)
2	8	
3		
4		

How is the last column related to dimension?

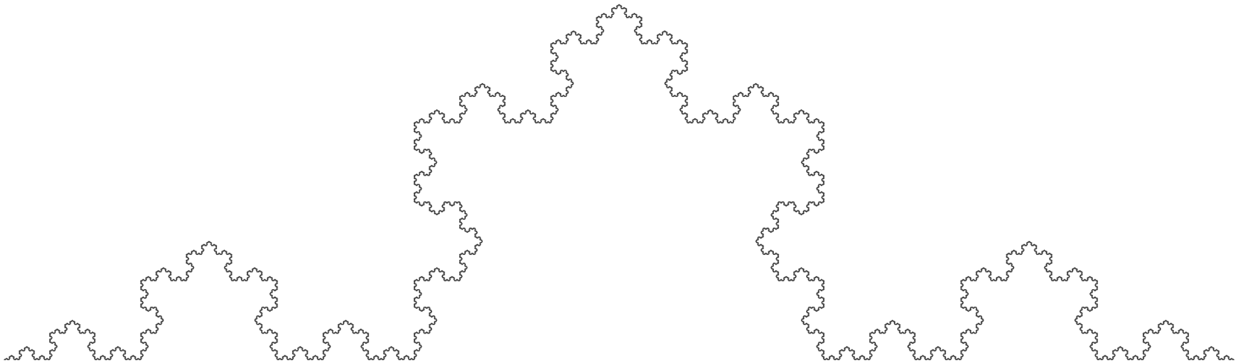
9. This fractal is called the *Sierpinski triangle*. If you scale it by 2 (as shown), how many times as big does it get? What is its dimension?



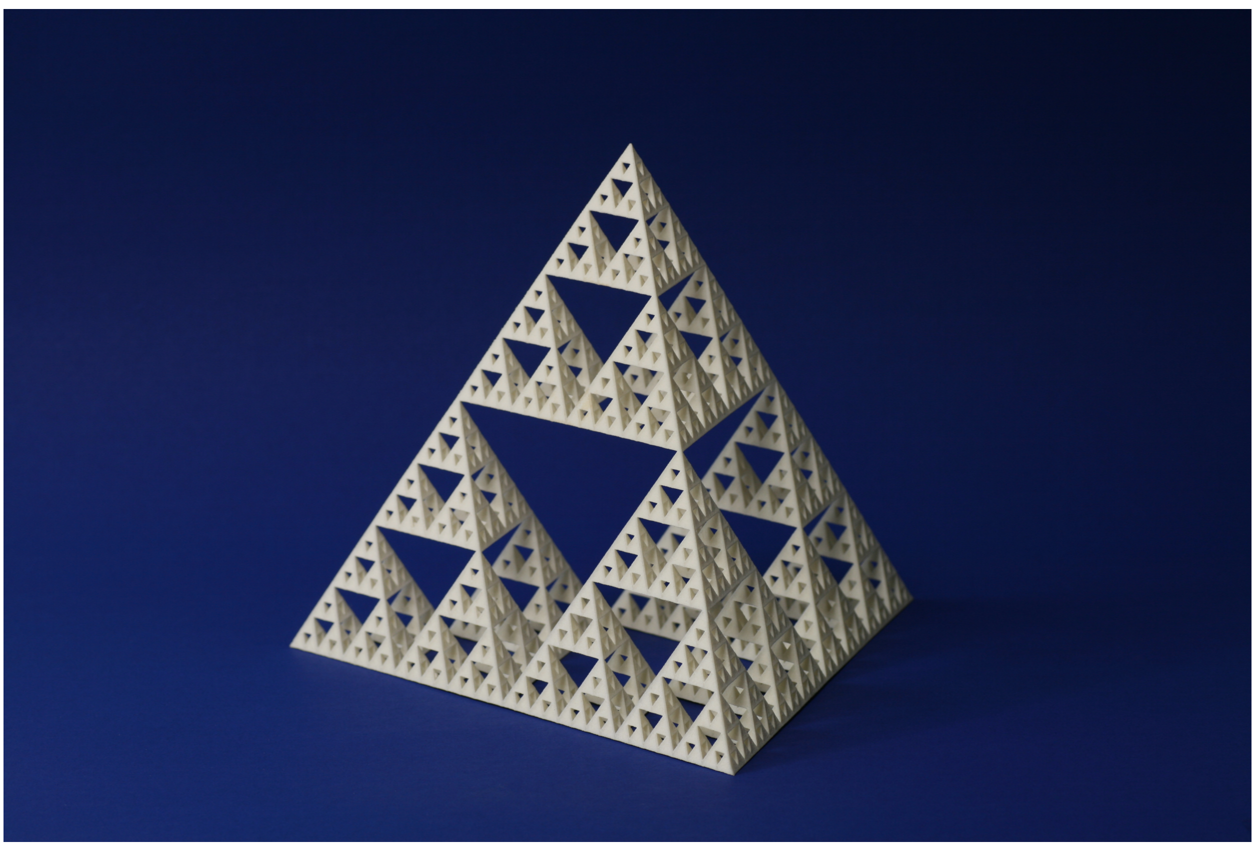
10. This fractal is called the *Sierpinski carpet*. What is its dimension?



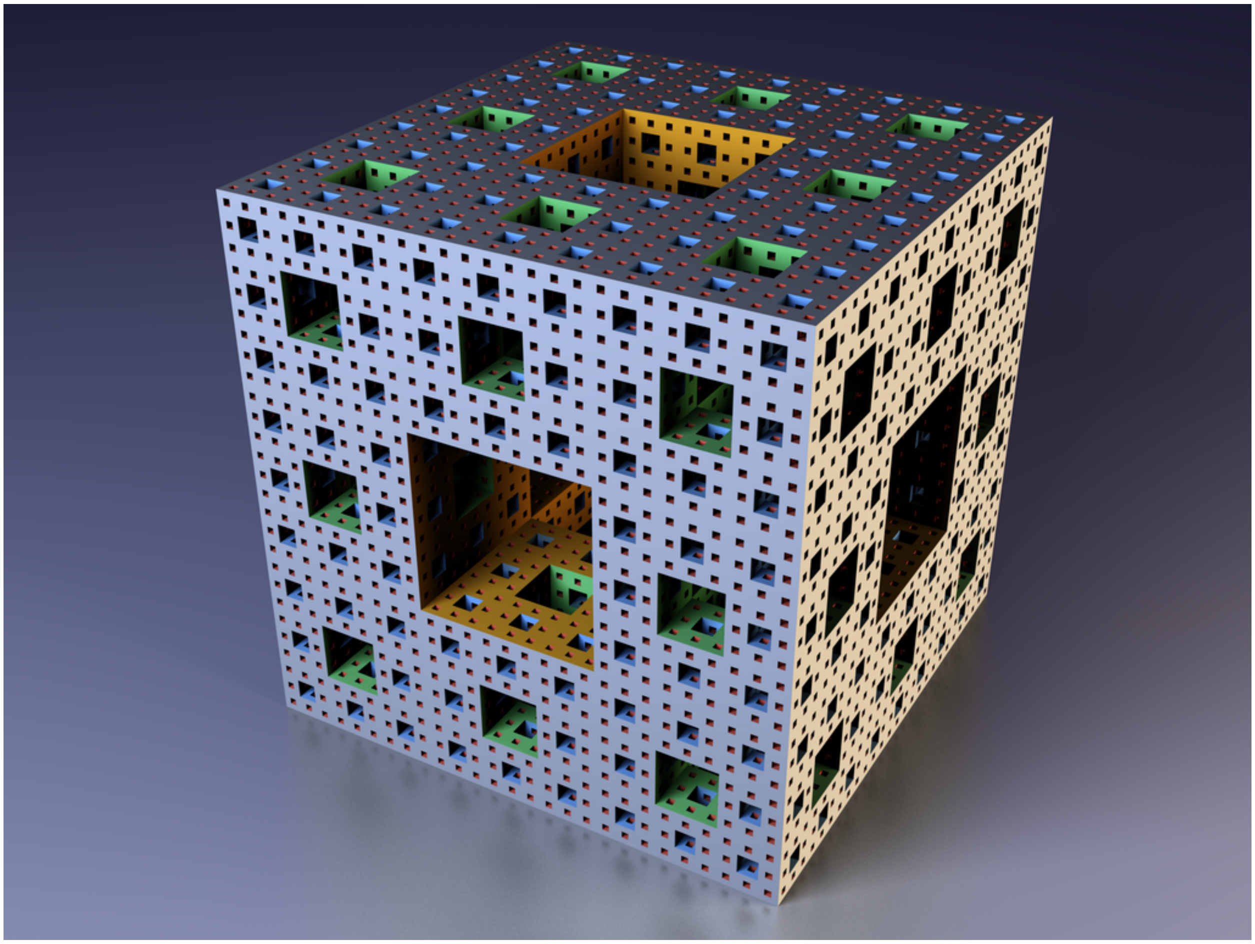
11. What dimension is one “side” of the Koch snowflake?



12. What dimension is the *Sierpinski tetrahedron*, pictured below?



13. What dimension is the *Menger sponge*, pictured below?



14. Can you draw a fractal with dimension $\log_4 5$?

15. Can you draw a fractal with dimension $\log_3 7$?

16. Can you draw a fractal with dimension $\log_2 5$?

17. Can you draw a fractal with dimension $\log_2 9$? Why or why not?