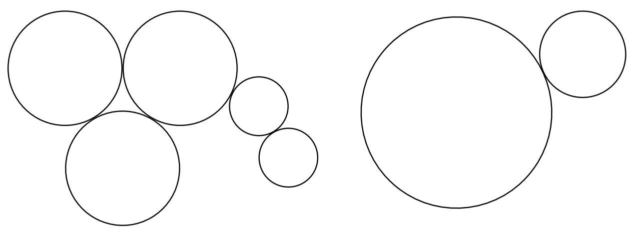
Name: ______________________________

UW Math Circle

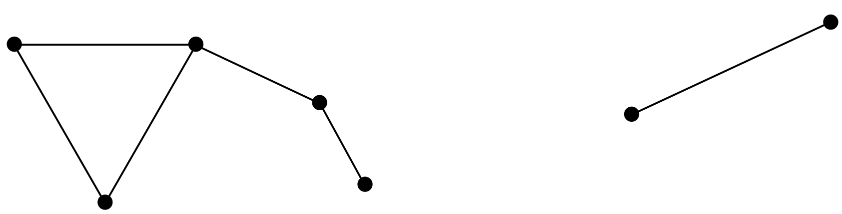
Week 17 - Circle Packings

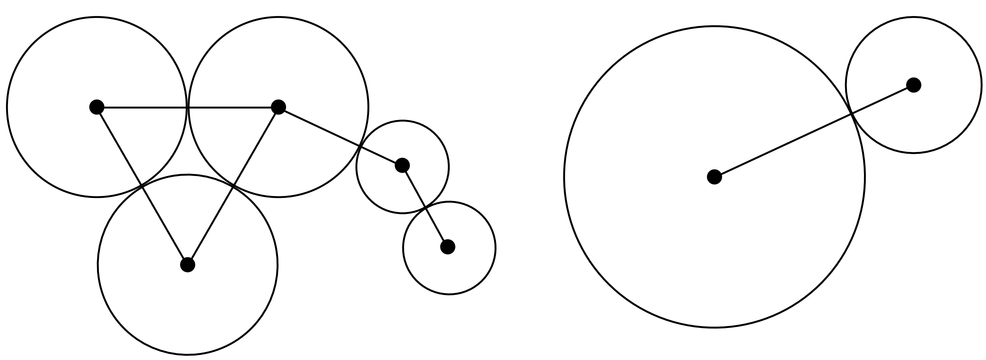
1 Circle Packings and Graphs

Today we will discuss "circle packings", which are placements of circles in the plane so that no two overlap. They are, however, allowed to touch; the picture below is an example of a circle packing.

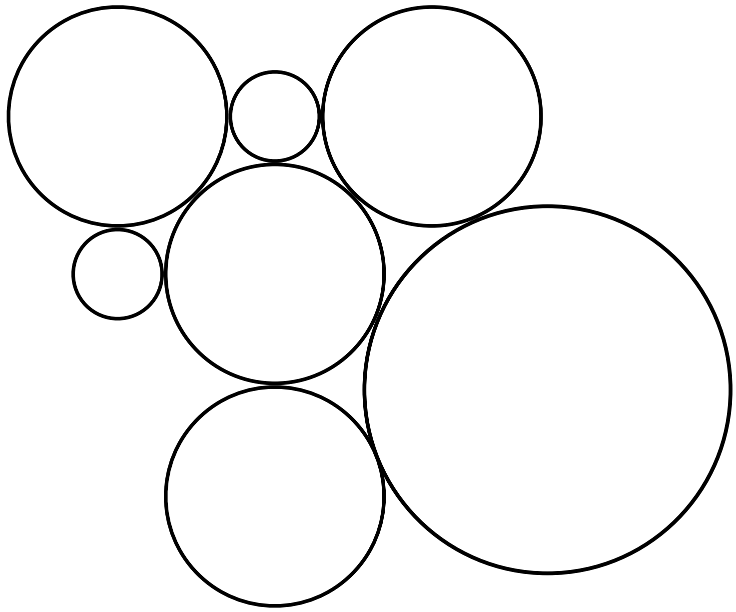


A *graph* is a collection of points, called vertices, and connections between two vertices, called edges. We can represent a circle packing with a graph by making a vertex for every circle and adding an edge between two vertices if the circles they correspond to touch. This is called the *contact graph* of the packing.

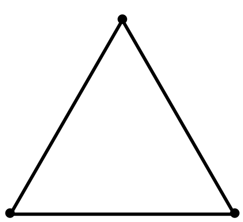




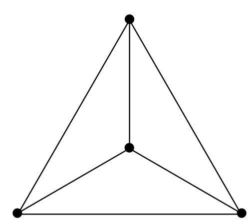
1. Draw the contact graph for the following circle packing



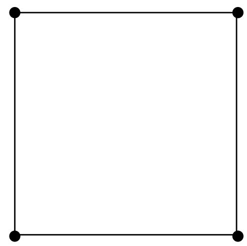
2. For each graph below, come up with a circle packing having the graph as its contact graph. Can you come up with more than one solution?

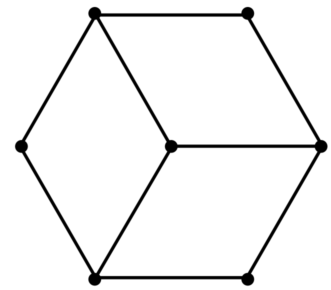


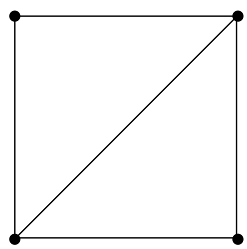
(a)

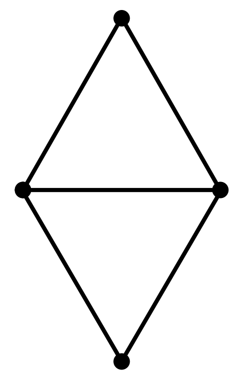


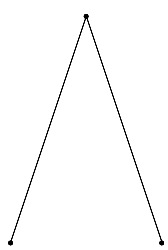
(b)

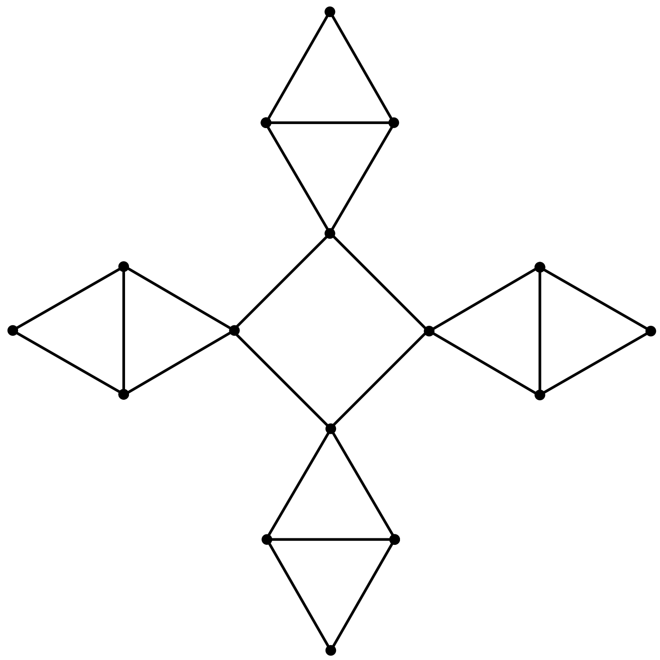
(c)


(f)


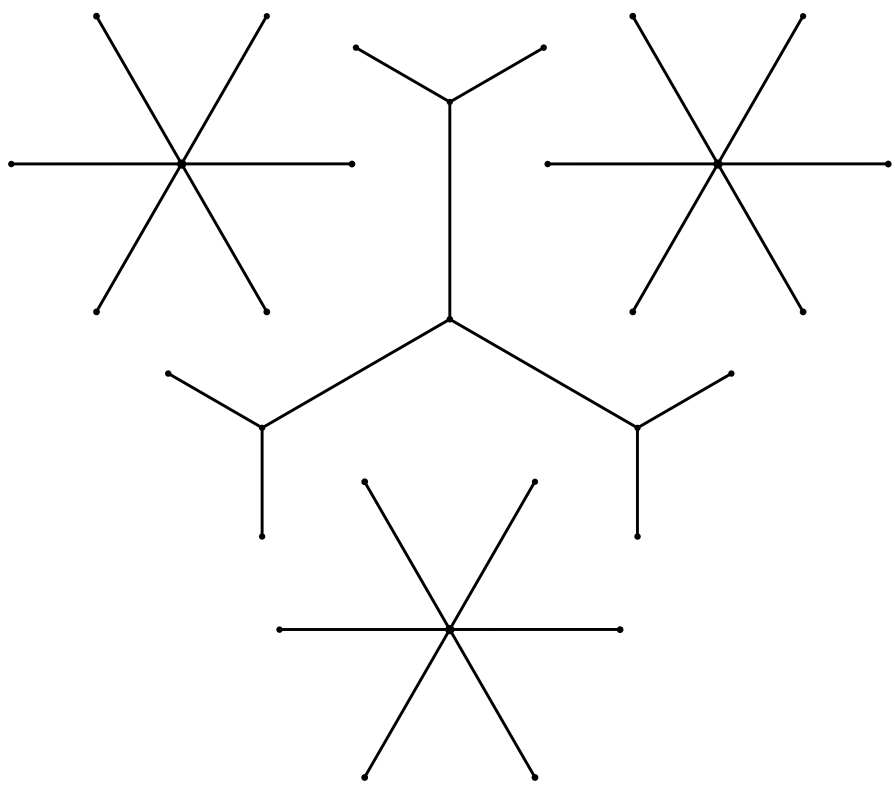
(d)



(g)

(e)


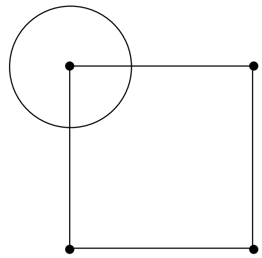


(h)



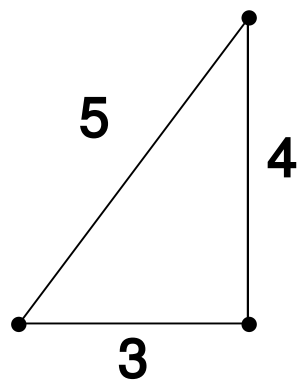
(i)

3. If you start with one circle already, how many ways are there to fill in the rest of the circles?



2 Triangles

4. Draw a circle packing with 3 circles for each of the following triangles.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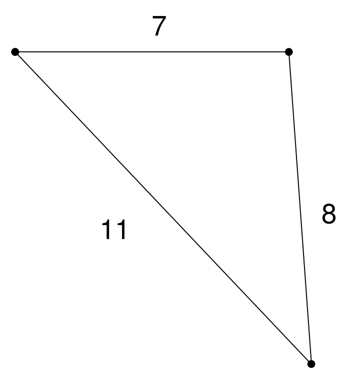
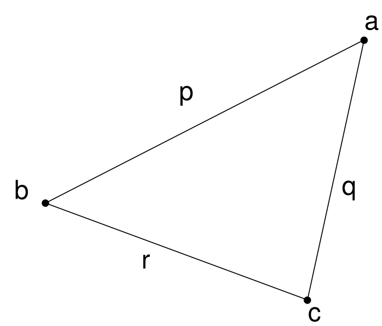
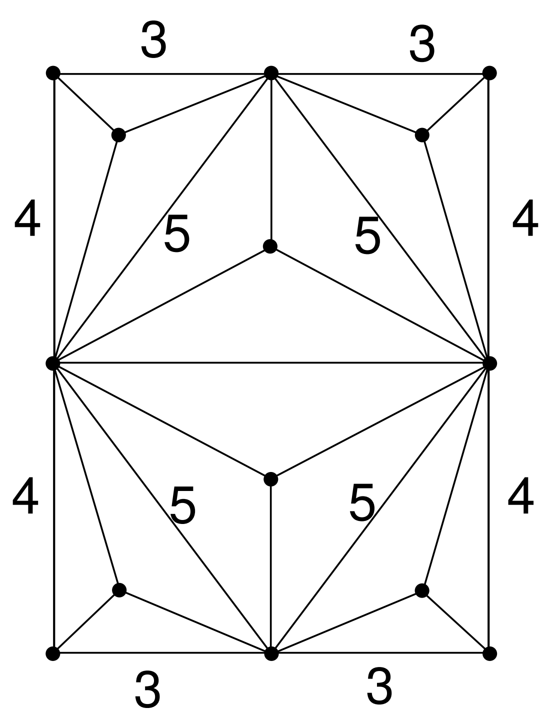

7
11
8

Triangles have a very special property as contact graphs; if you know the side lengths, there is exactly one circle packing having that triangle as its contact graph! If the side lengths are labeled as in the diagram below, then the circle centered at vertex a has radius $\frac{p+q-r}{2}$, the circle centered at vertex b has radius $\frac{p-q+r}{2}$, and the vertex centered at c has radius $\frac{-p+q+r}{2}$. In general, you can add the lengths of the two sides touching a vertex, subtract the side opposite, then divide by 2.



5. Use the formula above to make a circle packing corresponding to the following graph. Note we have only told you what some of the edge lengths are!



3 Regular Graphs

The *degree* of a vertex is the number of edges connected to it (incident to it). A graph is *regular* if every vertex has the same degree, often called k-regular where k is the common degree. For example, the triangle in $2(a)$ and square graphs in $2(c)$ are both 2-regular. The graph in $2(b)$ is 3-regular.

6. Draw a graph that is 2-regular and has 6 vertices.

7. Draw a 1-regular graph that has 6 vertices.

8. Can you draw a 1-regular graph that has 9 vertices?

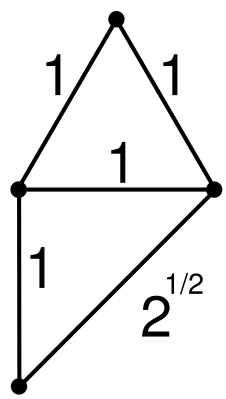
9. Draw 3-regular graphs with 4 vertices, with 8 vertices, and with 12 vertices.

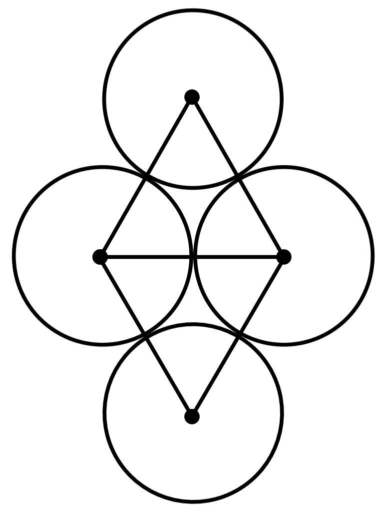
10. Are your graphs contact graphs of some circle packing? If not, can you change them so that they are?

11. Come up with a 3-regular graph that is the contact graph of a circle packing where all circles have the same radius.

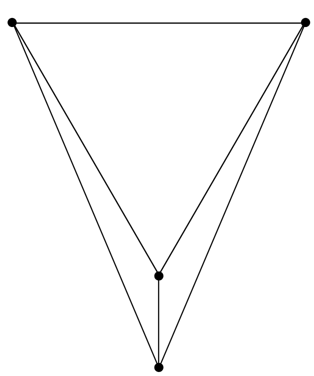
4 Repairing Graphs

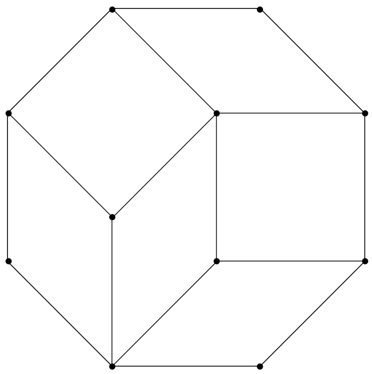
Now that you've explored circle packings and contact graphs, let's discuss the "circle packing theorem". This theorem states that any graph which can be drawn with no crossing edges is the contact graph of some circle packing, except you may need to move the vertices around (maintaining all the edges). For example, the graph drawn on the left is not the contact graph of any circle packing (why not?), but if we move the bottom vertex to the right place, we can make the contact graph of the right circle packing.





Repair the graphs below to create contact graphs and draw the corresponding circle packing.

12. 

13. 

14. Can you repair the following graph to be a contact graph for some circle packing?