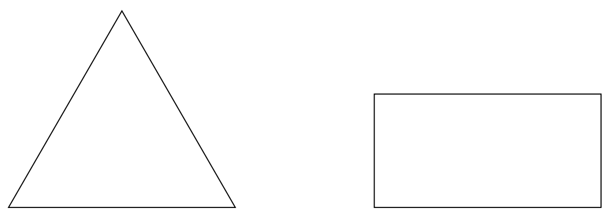
UW Math Circle

Week 9

1. It is now 5:00 pm, what time of day will it be 100 hours from now?
   Today is Tuesday, which day of the week will it be 100 days from now?

2. If you rotate the following triangle 900 degrees clockwise, what does it look like? If you rotate the following rectangle 900 degrees clockwise, what does it look like?



3. For each of the below equations, try to find nonnegative integers b and r that make the equation true. Make the integer r as small as possible. How does this relate to the previous problems?

   (a) $100 = b \cdot 7 + r$.

   (b) $100 = b \cdot 24 + r$.

   (c) $900 = b \cdot 120 + r$.

   (d) $900 = b \cdot 180 + r$.

4. For each of the below equations, try to find nonnegative integers b and r that make the equation true. Make the integer r as small as possible. Do you notice any patterns for r?

   (a) $2^2 = b \cdot 4 + r$.

   (b) $3^2 = b \cdot 4 + r$.

   (c) $4^2 = b \cdot 4 + r$.

   (d) $5^2 = b \cdot 4 + r$.

   (e) $6^2 = b \cdot 4 + r$.

5. For each of the below equalities, can you find integers n and b that makes the equality true? If not, argue that it is impossible.
   (Hint: think about the cases where n is even and n is odd).

   (a) $n^2 = b \cdot 4 + 0$.

   (b) $n^2 = b \cdot 4 + 1$.

   (c) $n^2 = b \cdot 4 + 2$.

   (d) $n^2 = b \cdot 4 + 3$.

Modular Arithmetic

As we see in the first few questions, sometimes we only care about the remainder when divided by a certain number, for example, the remainder of 100 divided by 24 is 4, so 100 hours from now is the same time as 4 hours from now. In these cases, we write

$$100 \equiv 4 \bmod 24.$$

There are many situations where we want to focus on the remainder, and in general, we write

$$a \equiv r \bmod n$$

when r is the remainder when a is divided by n.

1. For each number from 1 to 4 in the first column, try to find another number such that the product of them is 1 mod 5. For example, $3 \times 2 = 6$, and the remainder of $6 \div 5$ is 1, so $3 \times 2 \equiv 1 \bmod 5$.

number given	number you find	the product	remainder when divided by 5
1			
2			
3	2	$3 \times 2 = 6$	$6 \equiv 1 \bmod 5$
4			

2. Now, for each number from 1 to 5 in the first column, try to find another number such that the product of them is 1 mod 6. Can you find a number for every row? If not, argue why it is impossible for some numbers.

number given	number you find	the product	remainder when divided by 6
1			
2			
3			
4			
5			

3. Looking at your work in the mod 6 table, which numbers were "bad", meaning no matter what you multiply them by, you never get a remainder of 1? What do you notice about these numbers?

4. In mod 5, every number from 1 to 4 did have a partner that multiplied to give remainder 1. In mod 6, some numbers didn't. Why do you think this happens? What property of prime numbers might explain it?

Stamp Problem

1. You want to send a letter, but all your have are 4 and 5 cent stamps. Can you use these stamps to put the right amount of postage on your letter if the postage is 10 cents? 11 cents? 12 cents? What is the largest amount of postage that you cannot make with 4 and 5 cent stamps?

2. Now you have 3 and 5 cent stamps. What is the largest amount about postage you cannot make? What about with 7 and 11 cent stamps?

3. Formulate a conjecture for the maximum amount of postage you can't make with a and b cent stamps, where a and b are positive and their greatest common divisor is 1. Can you prove your conjecture is correct?

4. McDonald's chicken nuggets were sold in packs of 6, 9, and 20 pieces, what is the largest number of chicken nuggets that you cannot buy?

Sums of Squares

1. For each of the equalities below, try to find positive integers a and b to make the equality true.

   (a) $5 = a^2 + b^2$.

   (b) $13 = a^2 + b^2$.

   (c) $50 = a^2 + b^2$.

   (d) $74 = a^2 + b^2$.

2. If there are integers a and b so that $m = a^2 + b^2$ we say m is the **sum of two squares**. Try to decide if any of the following odd prime numbers is the sum of two squares (if so, write down which two squares sum to the number), and also write down their remainder when divided by 4.

odd prime	is it sum of two squares?	remainder when divided by 4
3		
5		
7		
11		
13		
17		
19		

3. Based on the previous problem, make a conjecture about which prime numbers can be written as the sum of two squares. Can you prove your conjecture?
(Hint: what did you find about n^2 mod 4 in problem 5 on page 2?)

4. (Challenge) Try to generalize your finding to composite numbers.