UW Math Circle

Week 6

Hackenbush is a combinatorial game played on a collection of line segments connected to each other at their endpoints or to a point on the "ground." Two players alternate turns "cutting" (deleting) line segments and removing any line segments now disconnected from the ground. The last player able to cut an edge wins.

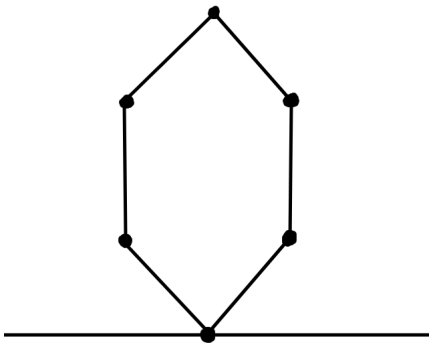
1. Hackenbush with a single branch is boring – let's play it with two! A game is determined by the lengths of the two branches. Fill each cell of the following table with a "1" if the first player has a winning strategy, or a "2" if the second player has a winning strategy:

First Branch Length

Second Branch Length	1	2	3	4	5
1					
2					
3					
4					
5					

- When does the first player always have a winning strategy? When does the second player always have a winning strategy?
- Describe a winning strategy for each player when possible, and prove that your strategy works for each player.
- Does it matter if the two branches are connected at the ground or not?

2. Let's play Hackenbush on loops with n edges! For example, a game of Hackenbush on a loop with 6 edges is played on the following arrangement of segments:



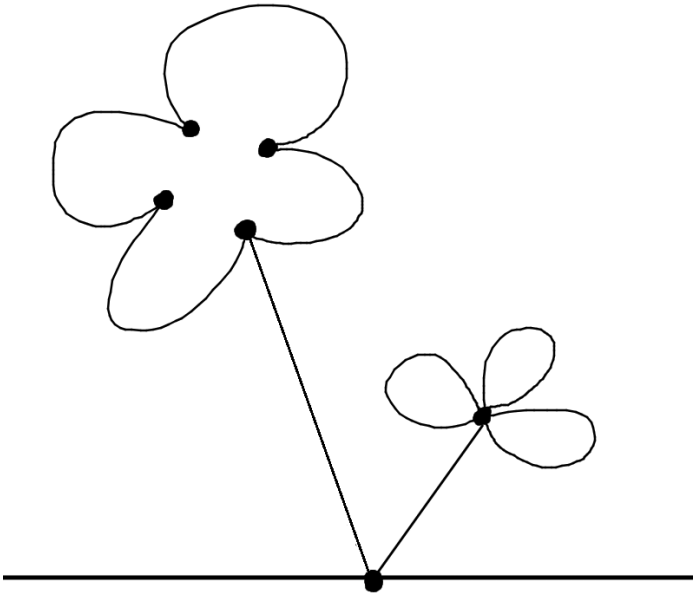
Fill out the following table with whether the first or second player to move has a winning strategy with two-branch Hackenbush:

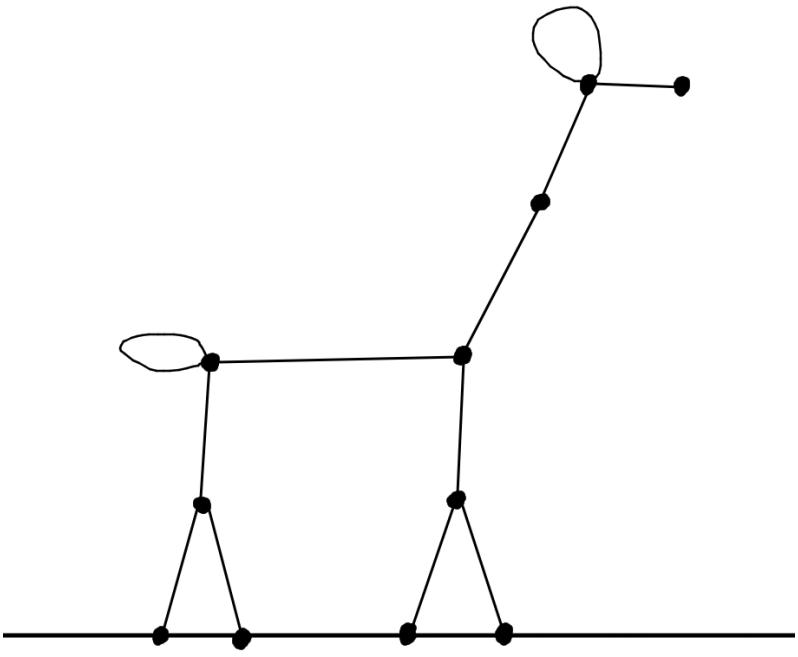
# of Edges in Loop

1	2	3	4	5	6	7

- When does the first player always have a winning strategy? When does the second player always have a winning strategy?
- Describe a winning strategy for each player when possible, and prove that your strategy works for each player.
- Does the result change if the two segments connecting to the ground touch the ground at different points?

3. Beat an instructor at the following games of Hackenbush. You get to pick whether you get to go first or second!





Addition Hacks

We can figure out which player will win any game of Hackenbush and compute the winning move at every step! But first, we'll need to learn about *nim sums*.

First, we'll want to write down positive integers in *binary*, or in base 2. In binary, numbers are only allowed to have digits 0 or 1, with a 1 in a given position representing twice the value of a 1 immediately to its right. For instance, the number 25 is represented in binary as:

$$13 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2 + 1 \cdot 1 = 1101_2$$

(We use the subscript 2 to indicate a number is written in binary.)

1. Write down the numbers from 1 to 10 in binary.

The *nim sum* of two integers a and b is denoted $a \oplus b$. To compute $a \oplus b$, first write down a and b in binary and line up the digits of a and b (in binary) in columns. Then, in the result, put 1s in the positions where the digits of a and b in that position differ, and 0s otherwise. Finally, read off the resulting binary number.

For example, the nim sum of 25 and 20 is 13. Note that $25 = 11001_2$ and $20 = 10100_2$, and doing the above process gives $1101_2 = 13$:

$$\begin{array}{rccccc} & 1 & 1 & 0 & 0 & 1 \\ \oplus & 1 & 0 & 1 & 0 & 0 \\ \hline & 0 & 1 & 1 & 0 & 1 \end{array}$$

2. Fill in every cell in the following table with the nim sum of the numbers at the top of its column and at the beginning of its row:

$a \oplus b$	1	2	3	4	5
1					
2					
3					
4					
5					

3. Prove that for any integer a that $a \oplus a = 0$, and if we have two integers a and b such that $a \oplus b = 0$, then $a = b$.

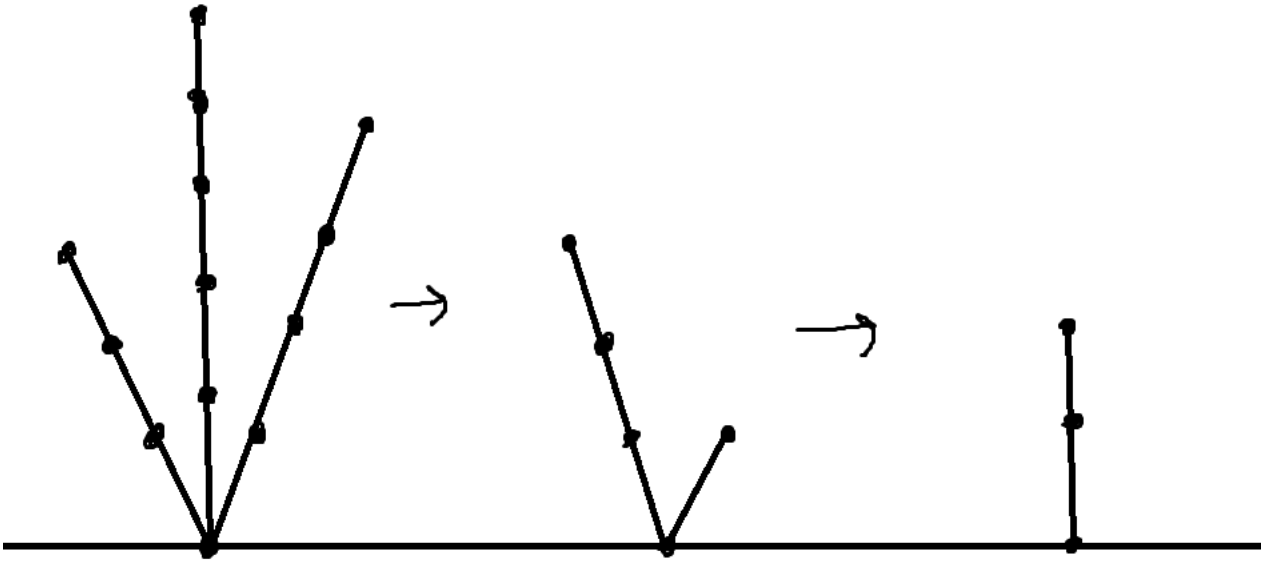
The nim sum helps us simplify games of Hackenbush:

Nim Sum Rule

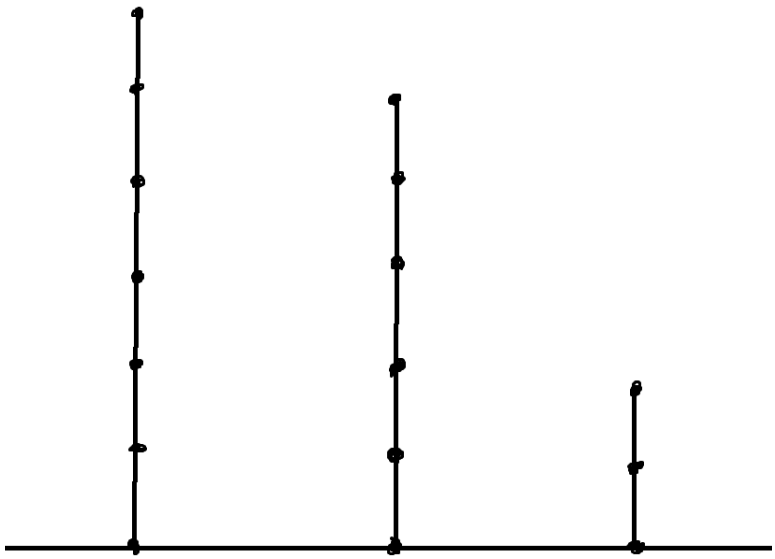
A game of two-branch Hackenbush with branches containing a and b edges is equivalent to a game of one-branch Hackenbush with $a \oplus b$ edges.

4. According to the Nim Sum Rule, when does the first player win? When does the second player win? Does this agree with your earlier analysis of two-branch Hackenbush?

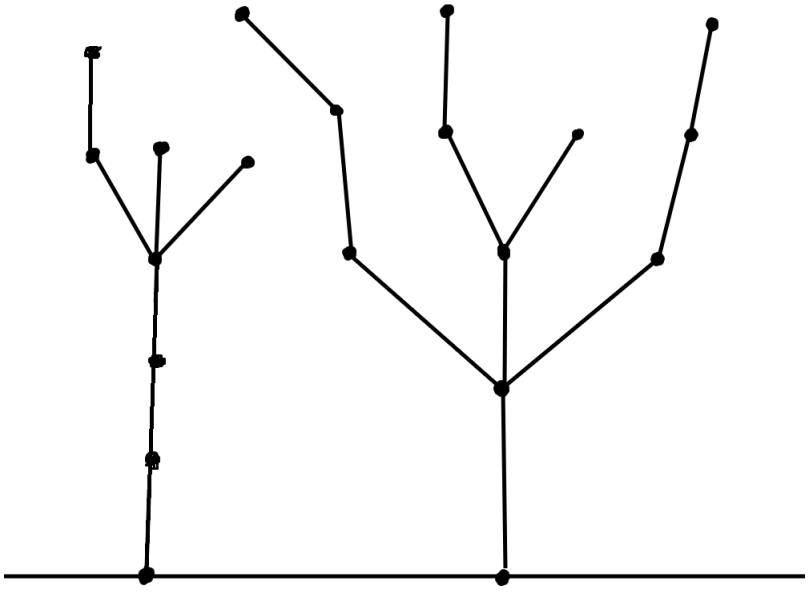
We can use the Nim Sum Rule to simplify places where two or more branches diverge into a single branch. For example, since the nim sum $(4 \oplus 5) \oplus 3 = 1 \oplus 3 = 2$, the following three branches collapse into a single branch:

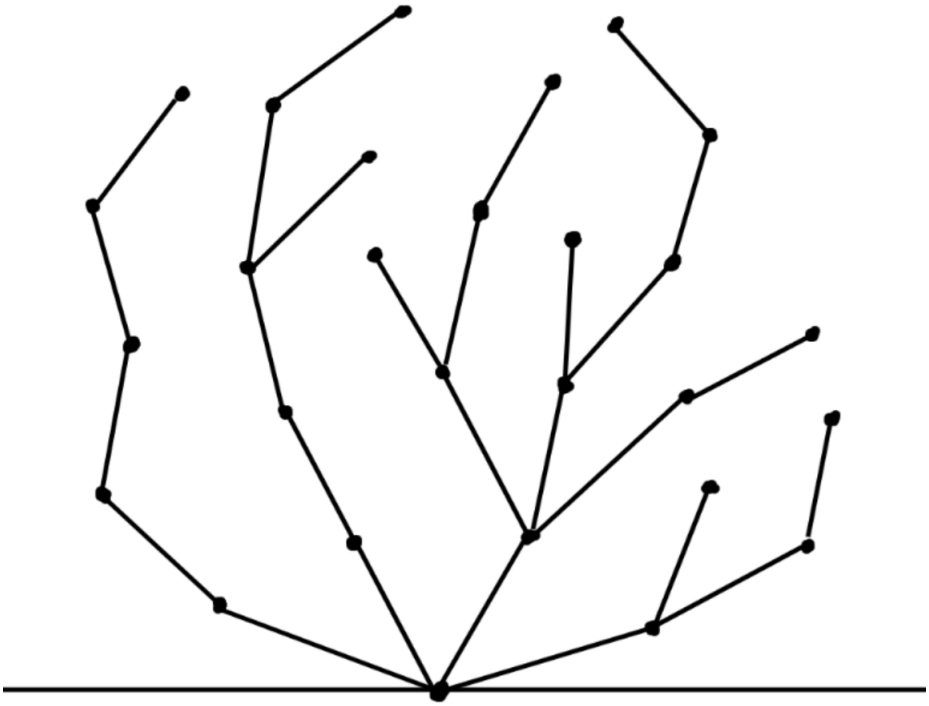


5. Simplify the following three branches into a single branch:

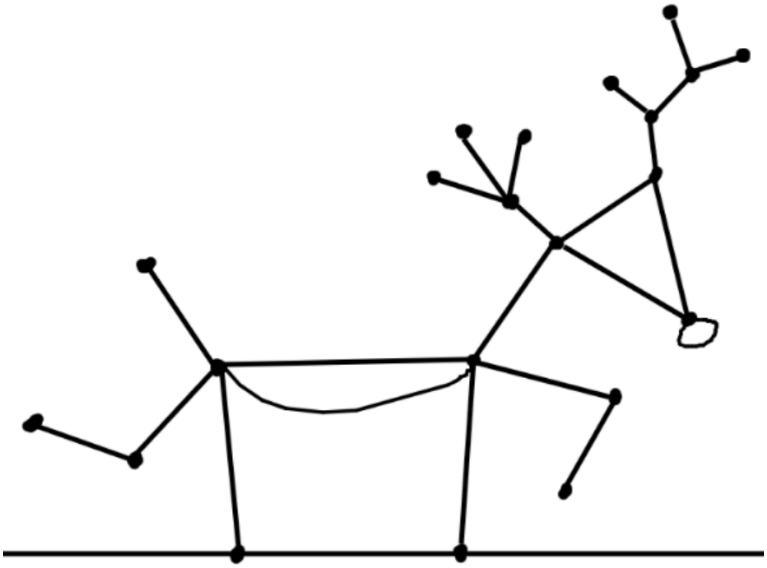


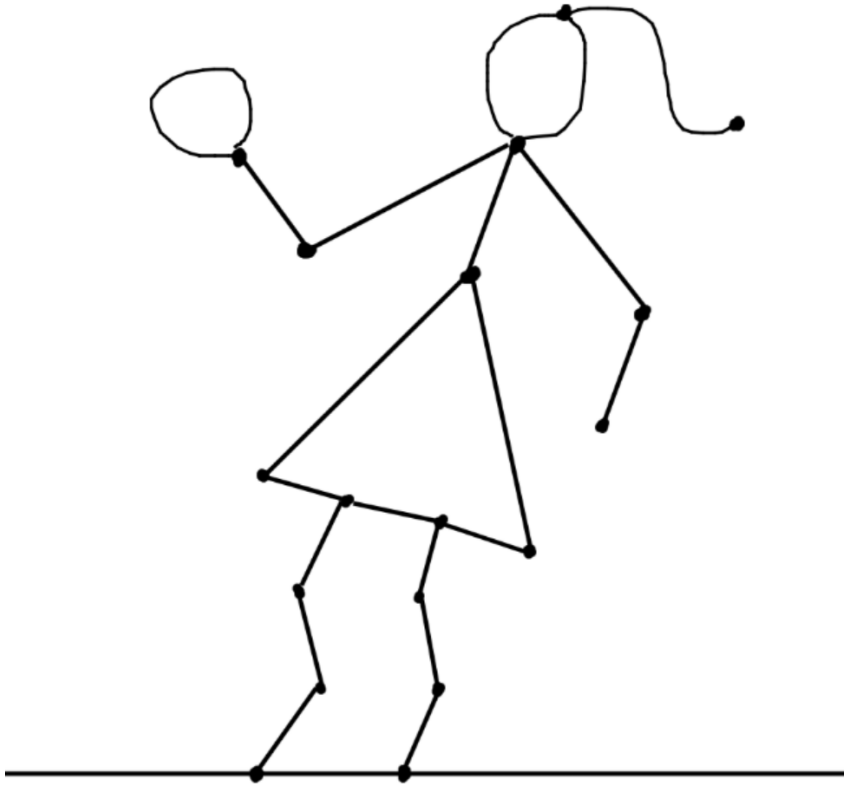
6. Use the Nim Sum Rule to simplify the following setups, and decide whether the first or second player has a winning strategy:





7. Defeat an instructor at Hackenbush on the following setups (again, you get to pick whether you go first or second)!

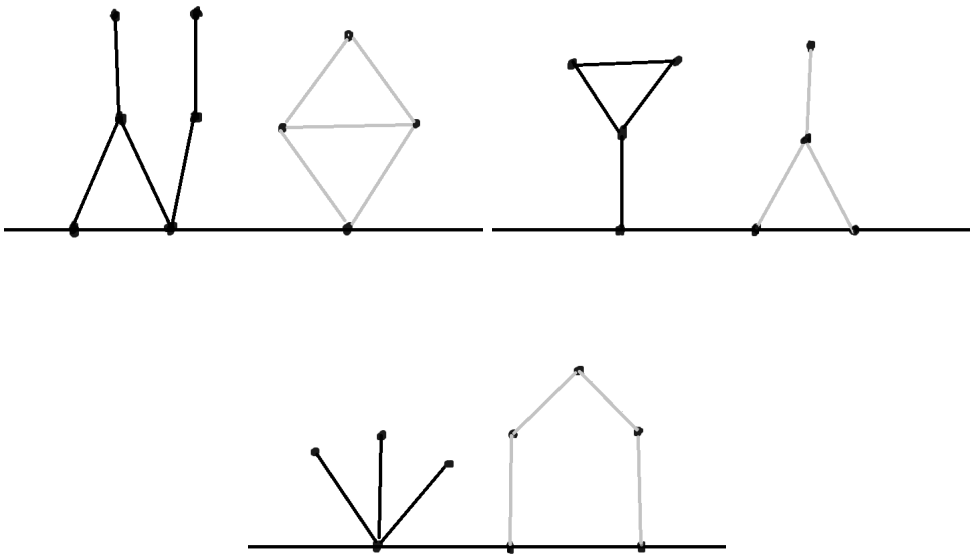




Now in Two Colors!

Now, color each segment in your games of Hackenbush one of two colors (here solid vs. gray), and assign each player to one of these colors. Now, the "solid" player can *only* cut solid segments, and similarly the "gray" player can *only* cut gray segments.
Let's first investigate games where every connected collection of edges is the same color:

1. Determine whether the solid player, gray player, first player, or second player has a winning strategy in the following games:



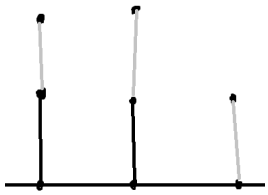
We assign games where every connected collection of edges is the same color an integer by taking (# of solid edges) − (# of gray edges). We call this the *game value* of a game of Hackenbush.

2. Compute the game values of each of the games above.

3. Which player (solid, gray, first, second) has a winning strategy when the game value is positive/negative/zero? Describe a winning strategy for this player in each case.

Now, let's try to find the game values of games where not every connected collection of edges is the same color.

4. Consider the Hackenbush game with a single gray edge on top of a single solid edge (which we'll call $*$): Play two $*$ games and a game with a single gray edge altogether:



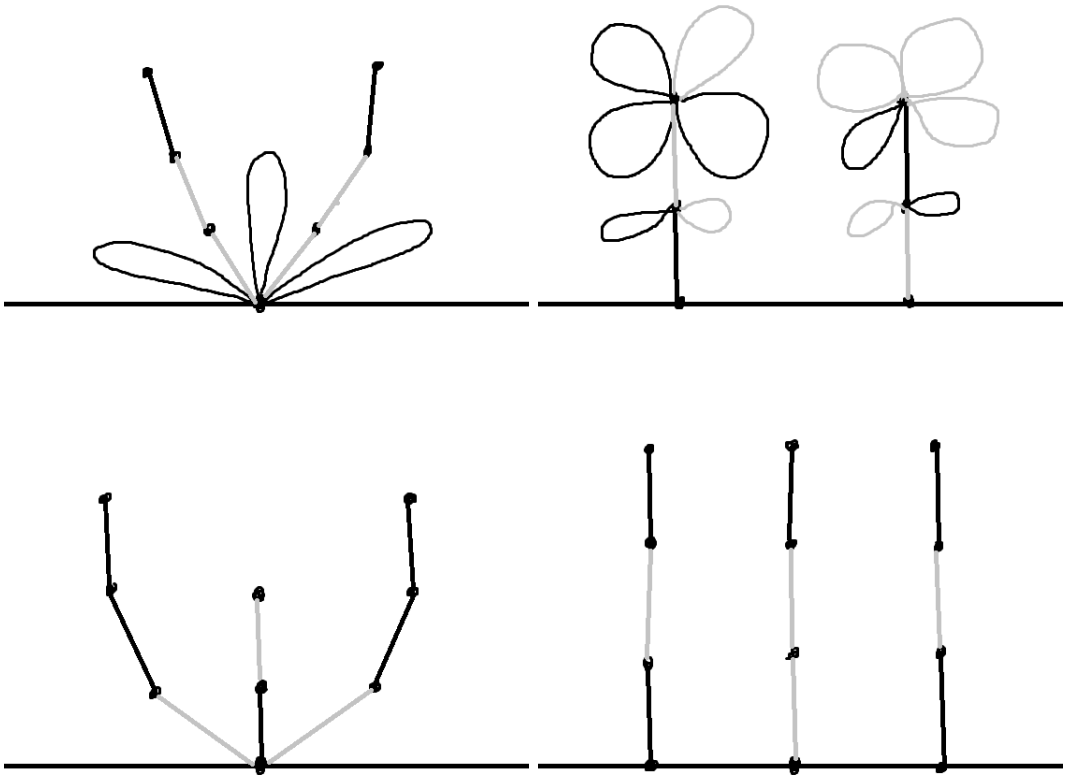
   Show that the second player has a winning strategy in this larger game, *regardless* of which color they are.

Note that game values are *additive* – if we play two games of Hackenbush at the same time, the total game's value is the sum of the two game values for the smaller games.

5. Recall what the value of a game where the second player has a winning strategy is. Use this to compute the game value of $*$. *You should get a fraction!*

6. One player has a clear advantage in $*$ – does the sign of your answer agree with your observations in Question 3?

7. Prove that for all of the following two-color Hackenbush games, the second player wins. Use this to compute as many new game values as you can!



In general, here's how to find the game value:

- The game value is a fraction with a power of two in the denominator (this can be 1).
- Any move made by the solid player should *decrease* the game value.
- Any move made by the gray player should *increase* the game value.
- If several different fractions work, pick the one with the *smallest* denominator!

7. Show that any values you calculated in Question 6 agree with this rule.

8. Beat an instructor at the following games of two-color Hackenbush. You get to decide whether you want to be the solid or gray player, and also whether you want to go first or second!