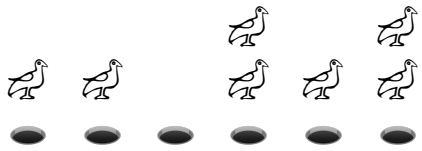
UW Math Circle

Week 5

The pigeonhole principle is a simple statement with profound implications. The reasoning goes like this:

Pigeonhole Principle

A room contains pigeons and pigeonholes. The pigeons each hide in a pigeonhole. If there are more pigeons than pigeonholes, there must be a hole containing at least two pigeons.



Ex: seven pigeons in six pigeonholes

As you solve each of these problems, describe what are the 'pigeons' and what are the 'holes,' and how many there are of each.

1. There is a drawer containing four different colors of socks. What is the least number of socks you need to take out to guarantee you have found a pair?

   Answer: You need to take out _____ socks.
   number

   There are _____ _____
   number "pigeons"

   and _____ _____,
   number "holes"

   so by the pigeonhole principle _____.

   If you only take out _____ socks, it is possible for none of them to be pairs.
   number

   An example where there are no pairs is _____

   _____.

2. Can you fill a 3×3 table with the numbers 0, 1, and 2 in such a way that the sums along the rows, columns, and diagonals are all different? (For example, in the table below, there are several sum that are 3 and several that are 4, so that configuration doesn't work.)

			⁄4
1	0	2	**3**
2	2	1	**5**
0	1	1	**2**
3	**3**	**4**	**\4**

Answer: There are _____ _____
number "pigeons"

and _____ _____,
number "holes"

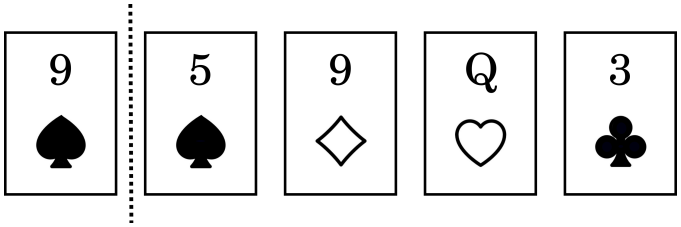
so by the pigeonhole principle _____.

3. Prove that out of any 11 integers, you can always find two whose difference is divisible by 10. [Hint: look at the 1s digits of each number.]

4. There are 22 Math Circlers in the class. Is it true that we can find four of them who were born on the same day of the week?

A Magic Trick

A magician gives a shuffled 52-card deck to their assistant who takes the first five cards. The assistant looks at the cards, chooses one, and reorders the remaining four cards. The assistant then reveals these four cards to the magician. After examining them, the magician correctly guesses the hidden card!

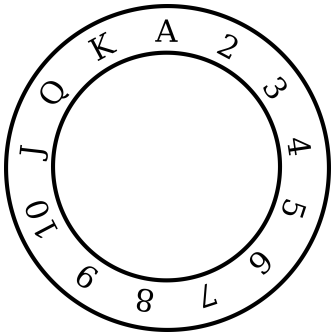


Ex: Magician sees four cards on the right and guesses the hidden card on the left

1. Prove that if the assistant chooses the card to keep **randomly**, there is no way to order the remaining four cards that would **uniquely** clue the fifth one. This means there is not enough information to guarantee the magician can guess correctly. [Hint: How many ways are there to order four cards?]

2. Follow the same set up as the initial trick, but this time the assistant is only allowed to show one of the remaining cards to the magician. Can you come up with a system so that the magician can always correctly guess the suit of the chosen card?

3. Below the possible ranks of cards are arranged in a circle. Prove that if you pick any two cards, one will be less than or equal to six spots to the right (clockwise) of the other.



4. (Challenge) Solve the original card trick! Can you come up with a system so that, based on the order that you show the remaining four cards, the magician can correctly guess the chosen card (both suit and rank)?

Numbers

1. There are 15 red balls and 15 green balls each marked with an integer between 1 and 100. Show there are at least two pairs consisting of one red and one green ball with the same sum.

2. Any two people at a dinner party are either friends or enemies. How many people must be at the party to guarantee that some group of three are all friends or all enemies?

3. Prove that in any list of 501 distinct integers between 1 and 1000, there must be two integers with one divisible by the other. Prove this is not always the case if you only have 500 of these numbers.

4. Your teacher hands you a list of 9 different numbers. Prove that you can always find three of them which are in order either from least to greatest, or greatest to least.

   For example, if your teacher hands you the numbers

   $$8 \quad 2 \quad 4 \quad 27 \quad 13 \quad 3 \quad 9 \quad 11 \quad 60$$

   then you could choose

   $$8 \quad 2 \quad \textcircled{4} \quad 27 \quad \textcircled{13} \quad 3 \quad 9 \quad 11 \quad \textcircled{60}$$

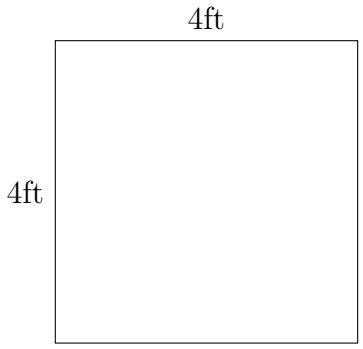
   or

   $$\textcircled{8} \quad 2 \quad \textcircled{4} \quad 27 \quad 13 \quad \textcircled{3} \quad 9 \quad 11 \quad 60$$

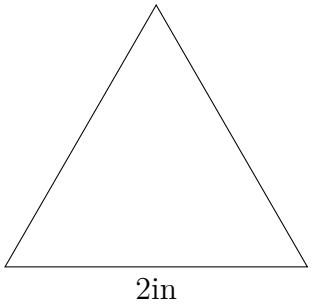
   Can you always find four with this property?

Geometry

1. The Brave Knight Cosmo rolls out dough into a 4-foot-by-4-foot square, then leaves to buy some sugar at the store. When he comes back, he sees that 33 flies have landed on the dough and are feeding on it! Angry, Cosmo reaches for his 1-foot-by-1-foot square baking pan. Show that he can bring the baking pan down on the dough to kill at least 3 flies with one blow.

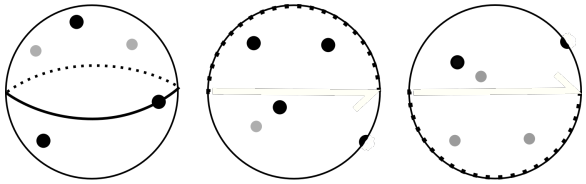


2. If there are five points inside an equilateral triangle of side length two inches, show that there must be two points within one inch of each other.



3. Remember from last week that a *polyhedron* is a closed 3-dimensional shape with flat sides. Prove that there cannot be a polyhedron where each face has a different number of edges.

4. A spherical ball has five dots on painted on its surface. Prove that there is a way to rotate the sphere so that at least four of the five dots can be seen from the top. Note: You can see half of the sphere from the top. If a dot lies exactly on the equator, you can still see it from the top.



Chess Puzzles

1. What is the greatest number of rooks you can place on an 8×8 chessboard so that no two of them attack each other?

2. What is the greatest number of bishops you can place on an 8×8 chessboard so that no two of them attack each other?

3. What is the greatest number of queens you can place on an 8×8 chessboard so that no two of them attack each other?

4. 64 whole numbers are written on the squares of a chessboard. The numbers in each row sum to an even number, as do the numbers in each column. Show that the sum of all the numbers on black squares is also even.