UW Math Circle

Week 7 – Musical Polygons

1 Polygons

If you have six points evenly spaced on a circle, you can draw a hexagon by putting your pencil on one, then drawing a straight line to the next one clockwise, then the next, repeating six times total, at which point you end up back where you started. If instead, you skipped the next closest one and drew a line to the point two over, you would get a triangle. If you started again on a point you haven't touched yet (say, 2), you would get two triangles.

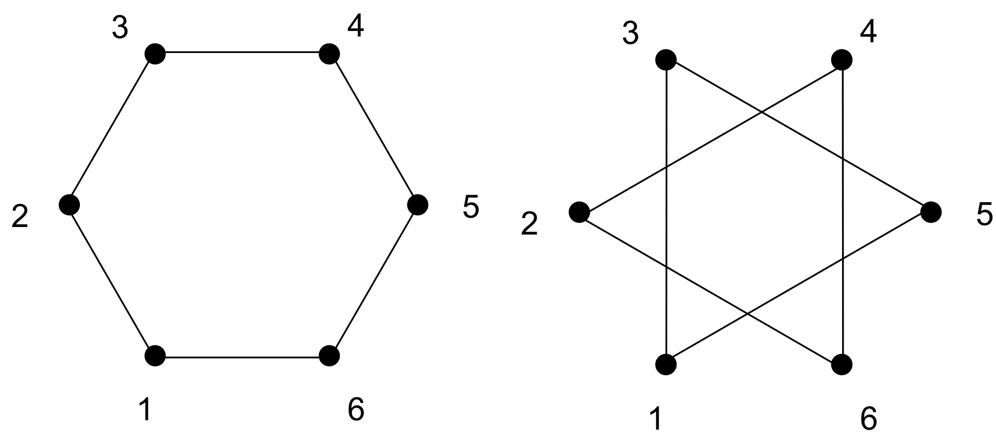

Figure 1: Two different drawings made by starting at 1 and moving one or two points clockwise.

1. Try this for a pentagon. What happens when you move 2? When you move 3? When you move 4?

2. What happens when you start with an octagon and move 2?

3. What happens when you start with a 9-gon and move 2? What about a 10-gon?

4. What happens when you start with an n-gon and move 2?

5. Start with a 10-gon. What happens when you move 3? How many points does the "star" you drew touch? How many times do you have to lift your pencil and start at a new point (always moving 3) before you've visited every point?

6. What happens when you start with an n-gon and move 3? How many n-gons

7. What happens when you start with an n-gon and move 4?

8. Start with a 12-gon. How many different stars do you have to draw to visit every point if you move 2? Do the same for every number from 2 to 11 (what happens if you move 12?). What do you notice?
   TIP: Think about writing 12 as $2 \cdot 2 \cdot 3$.

2 Intervals

The system most commonly used in Western music has 12 notes:

A, **A#/Bb**, B, C, **C#/Db**, D, **D#/Eb**, E, F, **F#/Gb**, G, **G#/Ab**

(The bold notes are the black keys on a piano.) After *G#/Ab*, the naming starts over at *A*, but this *A* is one "octave" higher than the first *A*. The difference between one note and the next in this system is called a "half step" or "semitone". In music, an "interval" is a pair of notes played together. A commonly used, widely considered pleasing interval is a "perfect fifth", which is any two notes where the higher note is 7 half steps above the lower note. Starting at *A*, the first perfect fifth is $A - E$, and the next fifth is $E - B$, then $E - F\#$, and so on. If you keep going up by fifths in this way, you will eventually play every note (albeit in different octaves) and then end up back at *A*! The notes appear in a different order from the order written above; this order is called the "Circle of Fifths".

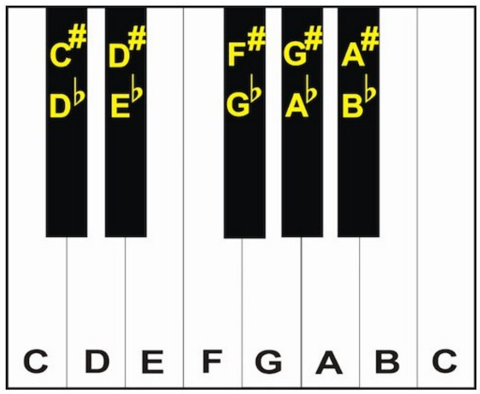

Figure 2: The 12 notes of the Western scale.

1. Try this out yourself. Starting at *A*, write out the next note a fifth above and keep going until you get back to *A*. How many times do you have to go up a fifth to get back to *A*?

2. Why does this happen? Can you explain this in terms of the polygons you explored in the first part?

3. What other intervals will visit every note before coming back to where they started?

Interval	Half Steps
Unison	0
Minor Second	1
Major Second	2
Minor Third	3
Major Third	4
Perfect Fourth	5
Augmented Fourth/Diminished Fifth	6
Perfect Fifth	7
Augmented Fifth/Minor Sixth	8
Major Sixth/Diminished Seventh	9
Minor Seventh	10
Major Seventh	11
Octave/Eighth	12

3 Strings

The sound of a string instrument comes from its strings vibrating. When a string vibrates, it produces sound waves with a *wavelength* equal to the length of the string. Longer wavelengths mean lower notes: so the longer a string is, the lower a note it makes.
On a violin or cello, you can play different notes by pressing down on a string, so only the part below your finger vibrates (effectively shortening the string). You can also make special notes called *harmonics* by touching a string lightly, so the parts above and below your finger vibrate at once!
When you split a string into two different lengths, the wavelength it vibrates at is the largest number that divides evenly into both lengths. For example, if you touch a one-meter string 2/5 m. from the top, it'll vibrate with wavelength 1/5 m., the biggest length that divides evenly into the top 2/5 m. segment *and* the bottom 3/5 m. segment.

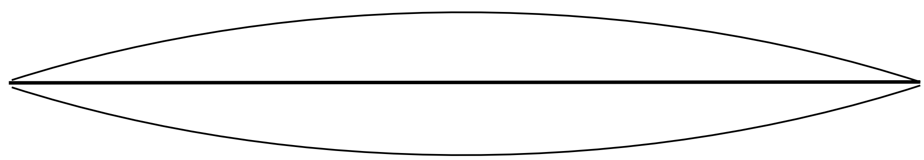

Figure 3: a string vibrating

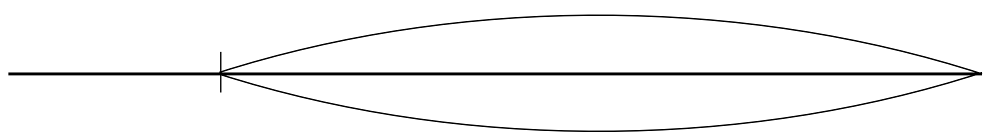

Figure 4: a string vibrating while pressed down

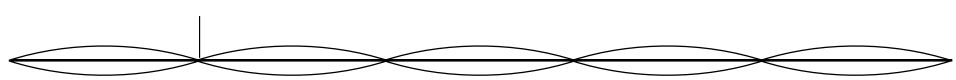

Figure 5: a string vibrating while touched (producing a harmonic)

1. What wavelength is the harmonic produced if the musician touches the string 1/4 m. down?
2. What wavelength is the harmonic produced if the musician touches the string 1/5 m. down?

3. What wavelength is the harmonic produced if the musician touches the string $1/n$ m. down?
4. What wavelength is the harmonic produced if the musician touches the string 2/4 m. down??
5. What about 3/4?
6. What about 4/10?
7. What about 6/10?
8. What about 3/12?
9. Can you relate this to the stars we drew in the polygons in the first part of the worksheet?

4 Integer Sums

Now we will explore something seemingly unrelated: sums of integers. After you've answered the questions, see if you can find a connection with the rest of today's activities.

1. What numbers you can make by adding a multiple of 5 and a multiple of 10?

2. What numbers you can make by adding a multiple of 2 and a multiple of 6?

3. What numbers you can make by adding a multiple of 2 and a multiple of 3?

4. What numbers you can make by adding a multiple of 12 and a multiple of 8?

5. If n and k are integers, what numbers you can make by adding a multiple of n and a multiple of k?

6. What numbers can you make by adding a multiple of 4, a multiple of 8, and a multiple of 10?

7. If p, q, r are integers, what numbers can you make by adding multiples of p, q, and r?