UW Math Circle

Week 4 – Fibonacci Numbers

The *Fibonacci sequence* is

$$\begin{array}{rcccccccccccc} & 1, & 1, & 2, & 3, & 5, & 8, & 13, & 21, & 34, & 55, & 89, & 144\ldots \\ \text{or} & F_1, & F_2, & F_3, & F_4, & F_5, & F_6, & F_7, & F_8, & F_9, & F_{10}, & F_{11}, & F_{12}\ldots \end{array}$$

The rule is that each number is the sum of the two previous numbers!

$$\begin{array}{lcl} F_3 = F_1 + F_2 & \text{because} & 2 = 1 + 1 \\ F_4 = F_2 + F_3 & \text{because} & 3 = 1 + 2 \\ F_5 = F_3 + F_4 & \text{because} & 5 = 2 + 3 \end{array} \qquad (\star)$$

We will call this the $(\star)$ rule. These numbers are full of fun patterns that we will explore!

Questions

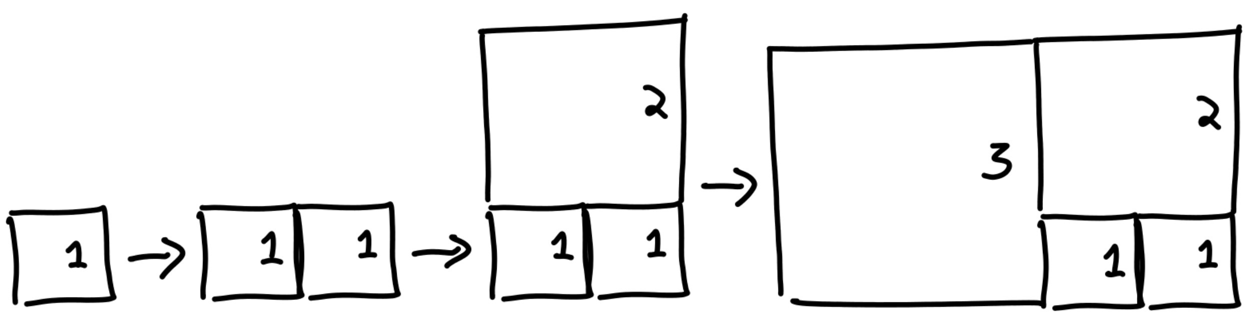
1. Here's how to draw a *golden rectangle*.

   Step 1: Draw a 1×1 square.

   Step 2: Attach a second 1×1 square.

   Step 3: Attach a 2×2 square to the long side of your rectangle from step 2.

   Step 4: Attach a 3×3 square to the long side of your rectangle from step 3.



   You can keep going: at each step, just attach a new square to the long side of your current rectangle. What will be the dimensions of your rectangle at step 11? *Hint: you don't actually need to draw all 11 steps!*

2. According to all known laws of bee genetics, a male bee always has one parent (a mother), while a female bee always has two parents (a mother and a father). Barry is a male bee. He has one parent (his mother), two grandparents (his mom's mother and father), and three great-grandparents (his grandpa's mother and his grandma's mother and father). How many great-great-great-great-great-great-great-great-grandparents does Barry have?

3. Here are the first few terms of the Icannobif sequence:

$$\begin{aligned} I_1 &= 1 \\ I_2 &= 1 \\ I_3 &= I_1 - I_2 = 0 \\ I_4 &= I_2 - I_3 = 1 \\ I_5 &= I_3 - I_4 = -1 \end{aligned}$$

What is I_{15}? *(Hint: you don't have to write out the entire sequence!)*

Showing patterns continue forever

The Fibonacci sequence is full of more fun patterns. For example:

$$\begin{aligned} &F_1 + F_2 = F_4 - 1 && \text{because} \quad 1 + 1 = 3 - 1 \\ &F_1 + F_2 + F_3 = F_5 - 1 && \text{because} \quad 1 + 1 + 2 = 5 - 1 \\ &F_1 + F_2 + F_3 + F_4 = F_6 - 1 && \text{because} \quad 1 + 1 + 2 + 3 = 8 - 1. \end{aligned}$$

It turns out this pattern continues forever! We can use our ($\star$) rule to reason

$$\begin{aligned} &F_1 + F_2 = F_4 - 1 && \text{because} \quad 1 + 1 = 3 - 1 \\ &F_1 + F_2 + F_3 = F_5 - 1 && \text{because} \quad (F_1 + F_2) + F_3 = F_3 + (F_4 - 1) = F_5 - 1 \\ &F_1 + F_2 + F_3 + F_4 = F_6 - 1 && \text{because} \quad (F_1 + F_2 + F_3) + F_4 = F_4 + (F_5 - 1) = F_6 - 1. \end{aligned}$$

And this reasoning can continue forever! See if you can reason why some of the following patterns will continue forever.

More Questions!

4. Notice the Fibonacci sequence

$$\begin{array}{lccccccccc} & 1, & 1, & 2, & 3, & 5, & 8, & 13, & 21, & 34, \ldots \\ \text{goes} & \text{odd,} & \text{odd,} & \text{even,} & \text{odd,} & \text{odd,} & \text{even,} & \text{odd,} & \text{odd,} & \text{even} \end{array}$$

Does this pattern go on forever? Why or why not?

5. Does the following pattern go on forever? Why or why not?

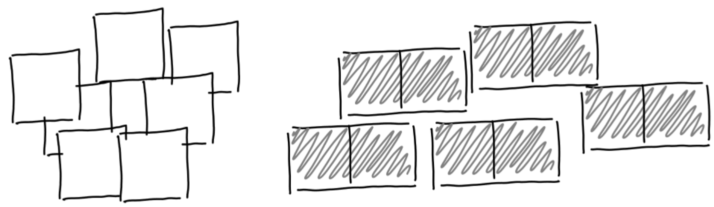
$$\begin{array}{lll} F_2 + F_4 = F_5 - 1 & \text{because} & 1 + 3 = 5 - 1 \\ F_2 + F_4 + F_6 = F_7 - 1 & \text{because} & 1 + 3 + 8 = 13 - 1 \\ F_2 + F_4 + F_6 + F_8 = F_9 - 1 & \text{because} & 1 + 3 + 8 + 21 = 34 - 1. \end{array}$$

6. Here's another fun pattern in the Fibonacci sequence:

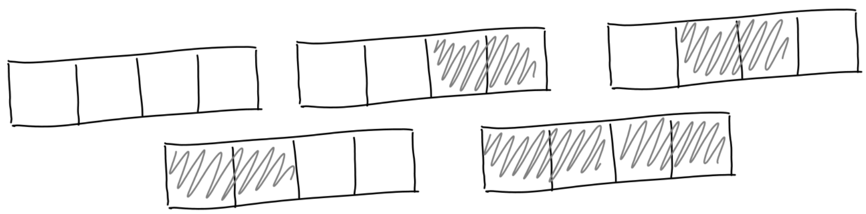
$$\begin{array}{lll} F_3 + F_5 = F_6 - 1 & \text{because} & 2 + 5 = 8 - 1 \\ F_3 + F_5 + F_7 = F_8 - 1 & \text{because} & 2 + 5 + 13 = 21 - 1 \\ F_3 + F_5 + F_7 + F_9 = F_{10} - 1 & \text{because} & 2 + 5 + 13 + 34 = 55 - 1. \end{array}$$

Does this pattern go on forever? Why or why not?

7. You have a pile of white 1×1 squares and grey 1×2 dominoes.



Notice you can use these squares and dominoes to fill up a 1×4 rectangle in 5 different ways:



How many ways can you use the squares and dominoes to

(a) Fill up a 1×1 square?

(b) Fill up a 1×2 rectangle?

(c) Fill up a 1×3 rectangle?

(d) Fill up a 1×4 rectangle?

(e) Fill up a 1×5 rectangle?

Do you notice a pattern? If so, does this pattern continue?

8. Does the following pattern continue forever? Why or why not?

$$\begin{aligned} 2F_2 &= F_3 + F_0 \quad \text{because} \quad 2 \cdot 2 = 3 + 1 \\ 2F_3 &= F_4 + F_1 \quad \text{because} \quad 2 \cdot 3 = 5 + 1 \\ 2F_4 &= F_5 + F_2 \quad \text{because} \quad 2 \cdot 5 = 8 + 2. \end{aligned}$$

9. Is it always true that $F_{m+n} = F_m F_n + F_{m-1} F_{n-1}$ for any *pair* of numbers m and n?

10. By problem 7, the Fibonacci number F_{n+1} is the number of ways to tile an $1 \times n$ rectangle with squares and dominoes.

    (a) How many ways are there to tile an $1 \times (n+1)$ rectangle while using at least one domino?

    (b) If you place dominoes and squares from left to right on the rectangle, how many ways are there to tile an $1 \times (n+1)$ rectangle where the last domino you place is covering squares $k+1$ and $k+2$?

    (c) Use (b) to prove there are $F_1 + F_2 + \cdots + F_n$ ways to tile a $1 \times (n+1)$ rectangle while using at least one square.

11. (a) In problem 10 you counted the number of ways to tile an $1 \times n$ rectangle in two different ways. Use this to give a different reason why the Fibonacci pattern we saw on the board is always true.

    (b) One way to show Fibonacci identities is to show that both sides of the equation are counting the same thing but in two different ways. Find counting problems that prove the patterns we saw in problems 5, 6, 8, and 9.