# UW Math Circle

## Week 2 – Mathematical Proofs

In mathematics, we want to be certain the things we claim to be true are true. If we can do this, our explanation of why is called a *proof.* A good proof starts with what we know for sure, and step by step builds up to what we want to prove.

The information we start with in a proof is called our *assumptions*. Each step should be small, follow from the previous step, and be 100% true. A step can use a definition, a fact we know, or a very small logical truth. The final step in our proof should confirm that our conclusion is true!

**Example:** We say a number $n$ is *even* if it can be written as $n = 2k$ for some integer $k$. We say $n$ is *odd* if it can be written as $n = 2h + 1$ for an integer $h$. Prove that if a number $n$ is even, then $n^2 + 1$ is odd.

*Proof.* Our assumption is that $n$ is even. By the definition of even, there is some integer $k$ where $n = 2k$. Then $n^2 + 1 = (2k)^2 + 1$ by substituting $2k$ for $n$. We can rewrite this using basic arithmetic:

$$\begin{aligned} n^2 + 1 &= (2k)^2 + 1 \\ &= 4k^2 + 1 \\ &= 2(2k^2) + 1 \end{aligned}$$

The number $2k^2$ is an integer, so by the definition of odd, $n^2 + 1$ is odd. $\square$

**Your turn:**

1. We wish to prove that if a number $n$ is odd, then $n^2$ is also odd.
   (a) What are your assumptions? What are you trying to prove?

   (b) Write a proof:

2. Your friend claims they can prove that $3 = 7$. What is wrong with their proof?

$$\begin{aligned} 7 &= 3 \\ 7 - 5 &= 3 - 5 && \text{(subtract 5 from both sides)} \\ 2 &= -2 \\ (2)^2 &= (-2)^2 && \text{(square both sides)} \\ 4 &= 4 \end{aligned}$$

This is true, so we have proven $7 = 3$.

3. The same friend shows you a magic trick. They tell you to pick a random 3-digit number. Then take the difference between it and its reverse. Add this number to its reverse. In the end, your number is 1089. You are amazed, and set out to prove this will always work.

   You claim to prove it as follows: Pick a random number, say 387. Subtracting from the reverse gives $783 - 387 = 396$. Adding this number with its reverse, you get $396 + 693 = 1089$. It worked with this random number, so it must be true.

   Does this prove the trick always works? Why or why not?

4. (BAMO, 2017) Consider the 4x4 "multiplication table" below.

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 2 | 4 | 6 | 8 |
| 3 | 6 | 9 | 12 |
| 4 | 8 | 12 | 16 |

   We create a path from the upper-left square to the lower-right square by always moving one cell either to the right or down. For example, here is one such possible path, with all the numbers along the path circled:

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 2 | 4 | 6 | 8 |
| 3 | 6 | 9 | 12 |
| 4 | 8 | 12 | 16 |

(a) What is the smallest sum we can possibly get when we add up the numbers along such a path? Prove your answer is correct.

(b) What is the largest sum we can possibly get when we add up the numbers along such a path? Prove your answer is correct.

## Proof by Cases

Sometimes when you are proving something, it is helpful to consider multiple scenarios. We call this a proof by *cases*. If you can prove a statement is true in in each case, then it must be true in general

WARNING: If you are splitting into cases, you have to consider all possibilities! If it is not clear that the cases you pick cover all possible scenarios, you must prove this.

**Example:** Alex, Beth, and Charlie are all wearing either a red or blue hat. Alex is wearing a blue hat. Charlie is wearing a red had. Alex is looking at Beth, and Beth is looking at Charlie. Is someone wearing a blue hat looking at someone wearing a red hat?

*Answer:* Yes. We know Beth must be wearing a red or a blue hat, so we split into these two cases.

CASE 1: Beth is wearing a red hat. In this case, by our assumption Alex wears a blue hat and is looking at Beth, who wears a red hat. Therefore someone in a blue hat is looking at someone in a red hat.

CASE 2: Beth is wearing a blue hat. By assumption, Beth is looking at Charlie who wears a red hat. Therefore someone in a blue hat is looking at someone in a red hat

In either case, the statement is true.

**Your turn:**

5. The cookie you were saving for lunch is gone. You ask your three friends, Amy, Bill, and Cole who ate it, and they tell you:

   Amy: I did not eat your cookie
   Bill: I did not eat your cookie
   Cole: Amy ate your cookie

   If only one person told the truth, who ate the cookie?

6. Suppose all octopuses have 6, 7, or 8 legs. Those with 7 legs always lie, while those with 6 or 8 legs always tell the truth. Four octopuses meet, count up all their legs, and state:

   Octopus 1: In total, we have 28 legs.
   Octopus 2: In total, we have 27 legs.
   Octopus 3: In total, we have 26 legs.
   Octopus 4: In total, we have 25 legs.

   How many legs do the octopuses each have?

7. A number is *prime* if it is only divisible by one and itself. Prove that every prime number larger than 3 is either one more or one less than a multiple of 6.

## Proof by Contradiction

So far the proofs we have done have been *direct proofs*, meaning we start with our assumptions and logically work toward our conclusion. Sometimes it is helpful to start a different way. We can prove something is true by assuming it is false, and then proving this is not possible by arriving at a conclusion we know to be false. This false conclusion is called a *contradiction*.

**Example:** A number is said to be *rational* if it can be expressed as a fraction of integers (such as 4/7). Otherwise, the number is *irrational*. Prove that $\sqrt{2}$ is irrational.

*Proof.* Assume for contradiction $\sqrt{2}$ is rational. Then there are integers $a$ and $b$ where $\frac{a}{b} = \sqrt{2}$. We can choose these $a$ and $b$ so that the fraction is fully reduced, meaning there is no number which divides both $a$ and $b$. Squaring both sides we get

$$\frac{a^2}{b^2} = 2$$
$$a^2 = 2b^2$$

By the definition of even, $a^2$ is even. By problem 1 on this worksheet, since $a^2$ is not odd, $a$ cannot be odd either. Thus $a$ is even, so $a = 2k$ for some integer $k$. Substituting this into our equation we get

$$(2k)^2 = 2b^2$$
$$4k^2 = 2b^2$$
$$2k^2 = b^2$$

Therefore $b^2$ is even, so $b$ is even. This means both $a$ and $b$ are divisible by 2, contradicting our earlier claim.

By contradiction, $\sqrt{2}$ is irrational. □

8. Prove that an irrational number plus a rational number is irrational.

9. There are 25 girls and 25 boys seated at a circular table. Prove that someone is seated between two girls.

10. If 5 points are placed within an equilateral triangle with side length 1, then you can always find two points that are within distance of $\frac{1}{2}$ from each other.

## More Problems

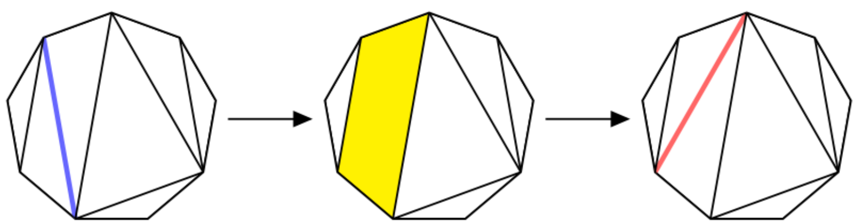
11. Freddy claim that $n^2 + n + 41$ is a prime number for all positive integers $n$. Is this true?

12. Answer (a) and (b) from question 4 but for an $n \times n$ multiplication table for any integer $n$.

13. Either prove the trick in problem 3 always works (with a correct proof) or give a counterexample. If it does not work for every three digit number, which numbers does it work for?

14. Logicians A, B and C each wear a hat with a positive integer on it. They are told that the number on one hat is the sum of the numbers on the other two. They can see the numbers on the other two hats but not their own. They are asked in turn if they can identify their number. In the first round A, B and C each say they don't know. In the second round A is first to go and states their number is 50. What numbers are on B and C?

15. (UW Math Hour Olympiad, 2019) A *triangulation of a regular polygon* is a way of drawing line segments between its vertices so that no two segments cross, and the interior of the polygon is divided into triangles. A flip move erases a line segment between two triangles, creating a quadrilateral, and replaces it with the opposite diagonal through that quadrilateral. This results in a new triangulation.



Given any two triangulations of a polygon, is it always possible to find a sequence of flip moves that transforms the first one into the second one?