Name: ______________________________

UW Math Circle

Week 14 – Dissections

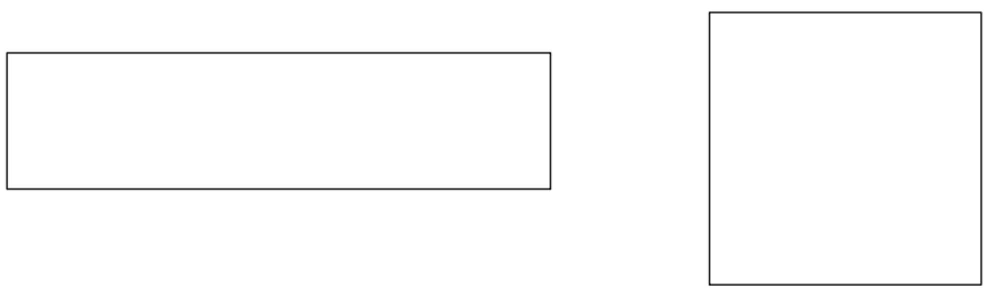
In today's circle, we will be cutting up shapes and rearranging them into other shapes!

1 Simple Dissections

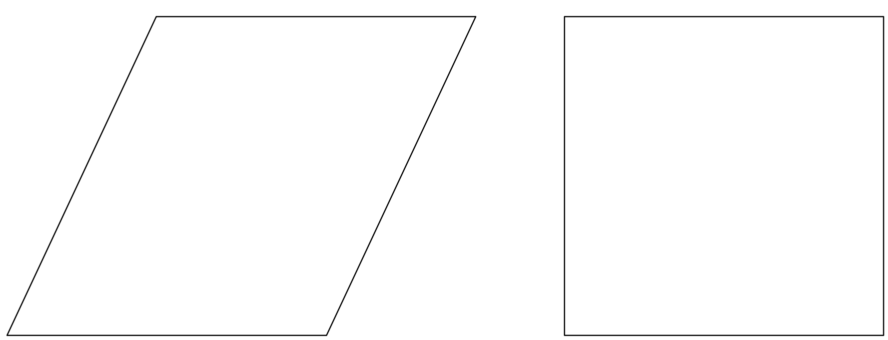
A *dissection* is a way of cutting up one or more shape into pieces and rearranging them to form other shapes. For all of the pairs of figures below, cut one up into pieces and rearrange it into the other. Try to cut the figures into as few pieces as possible.

TIP: Instead of cutting out the shapes, draw lines on one figure where you would cut, and draw lines on the other figure to show how the pieces fit together.

1. Can you cut the rectangle up into pieces to form the square on the right? Can you go backwards, from the square to the rectangle?



2. Can you cut the parallelogram up into pieces to form the square on the right? Can you go backwards?



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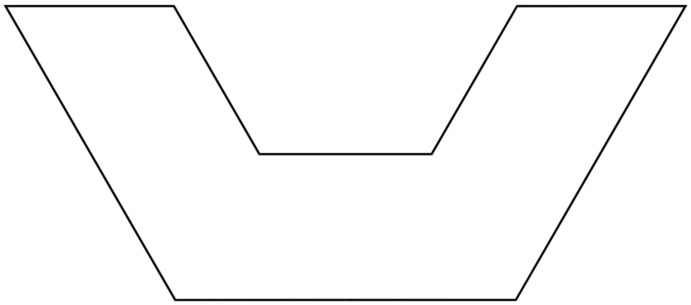

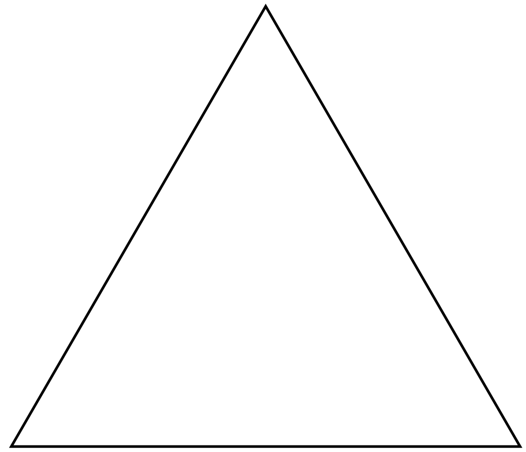

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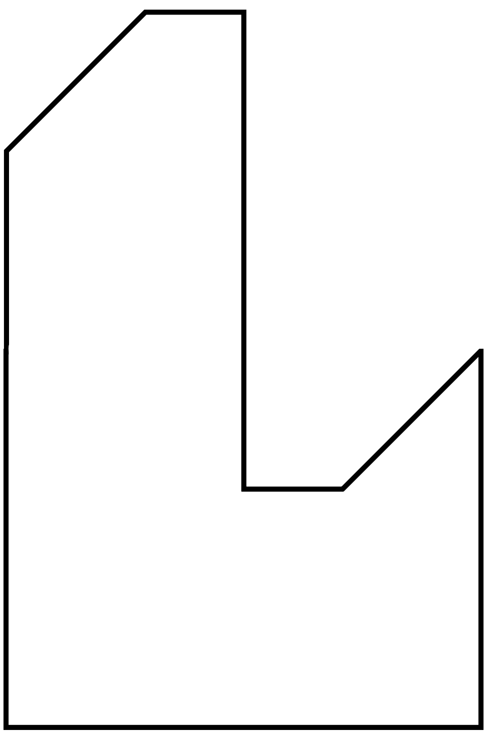
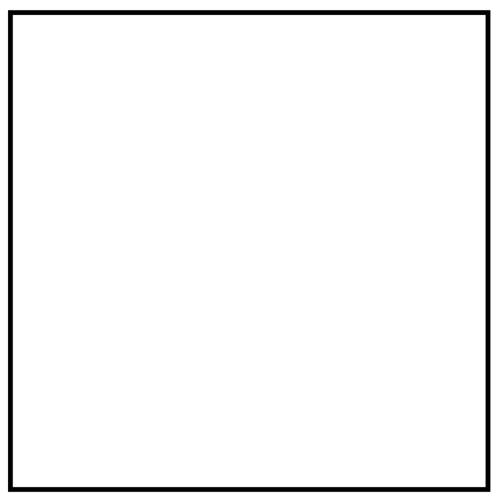
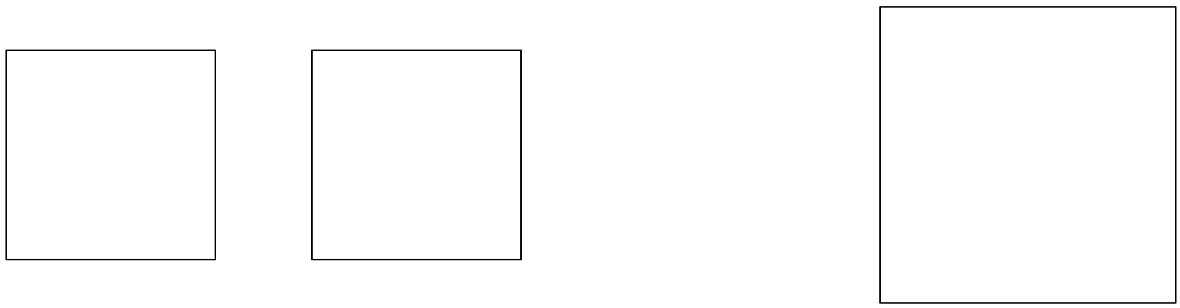
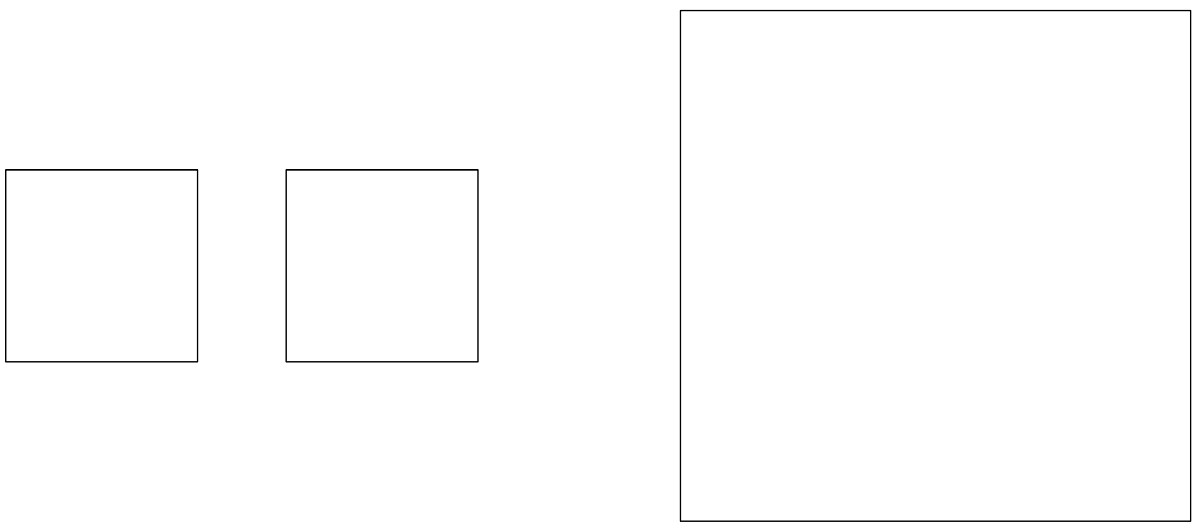




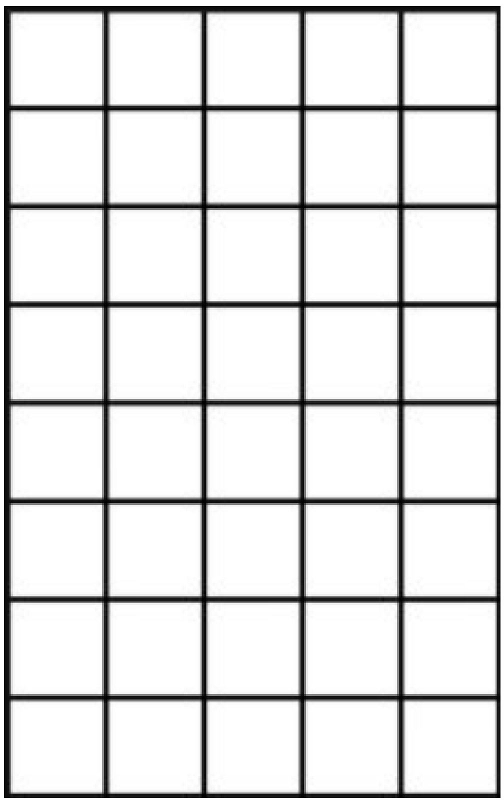
5. Can you turn the two squares on the left into the single larger square on the right? If so, how? If not, why not?

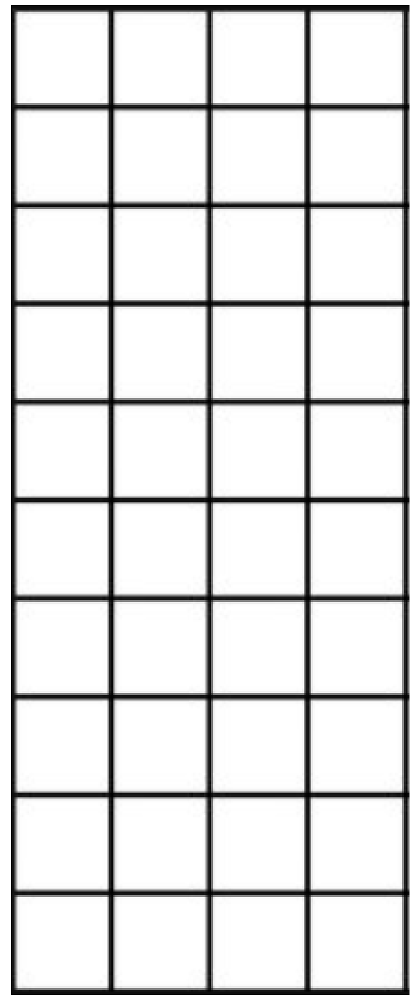


6. Can you turn the two squares on the left into the single larger square on the right? If so, how? If not, why not?



7. What is the fewest pieces you can cut a 5×8 rectangle into to be able to rearrange it into a 4×10 rectangle?





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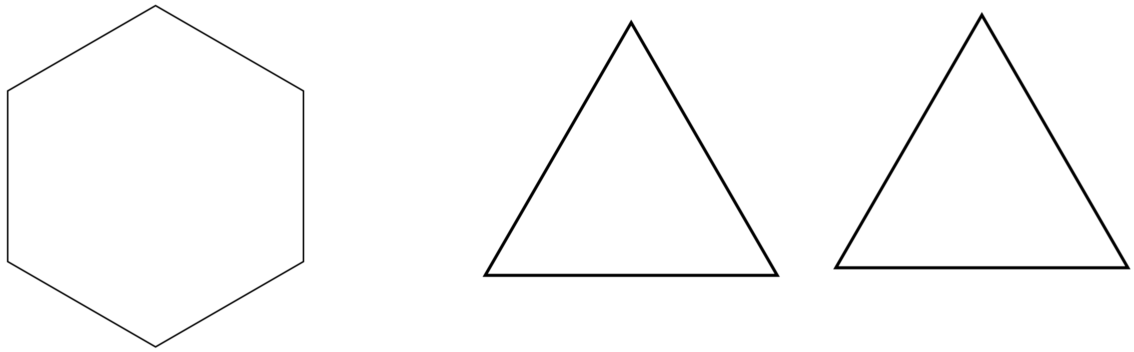

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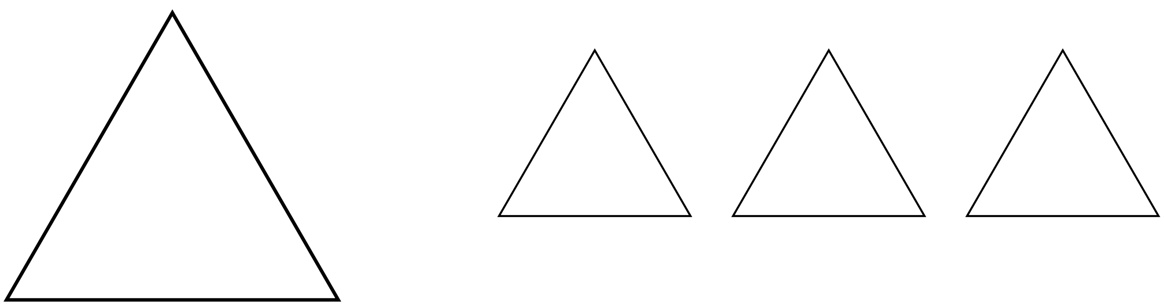

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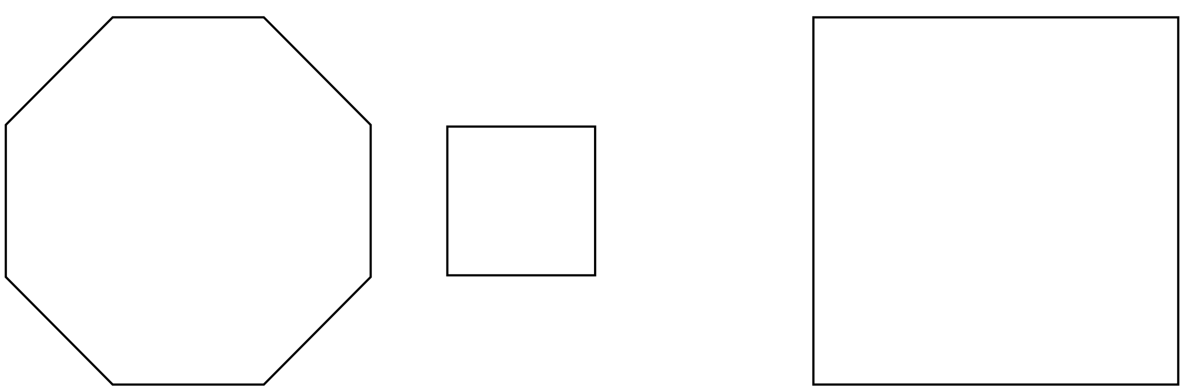
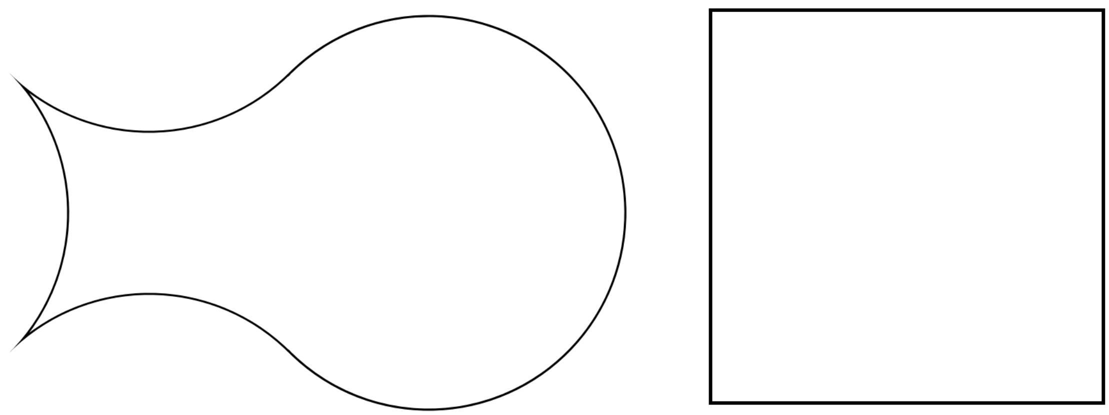
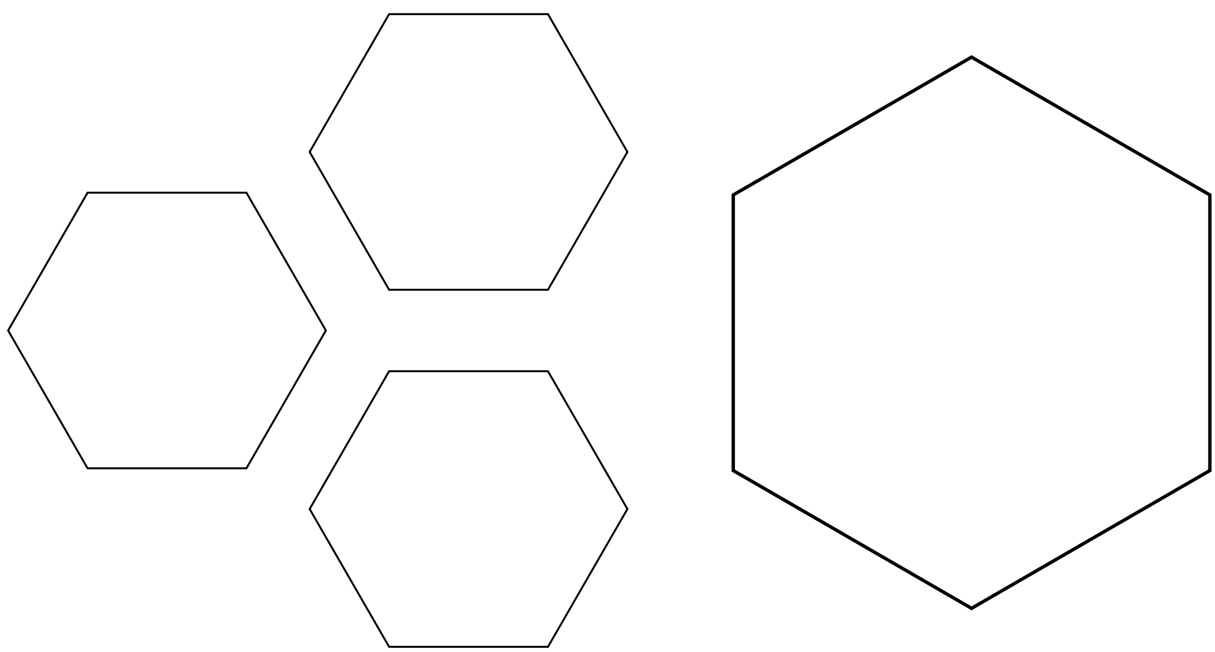


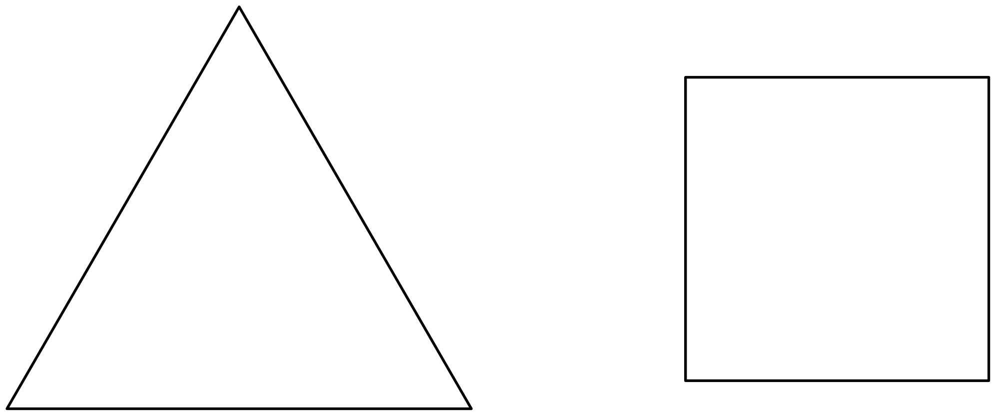
11. Can you dissect this “bulbfish” into a square using exactly three pieces?



12.



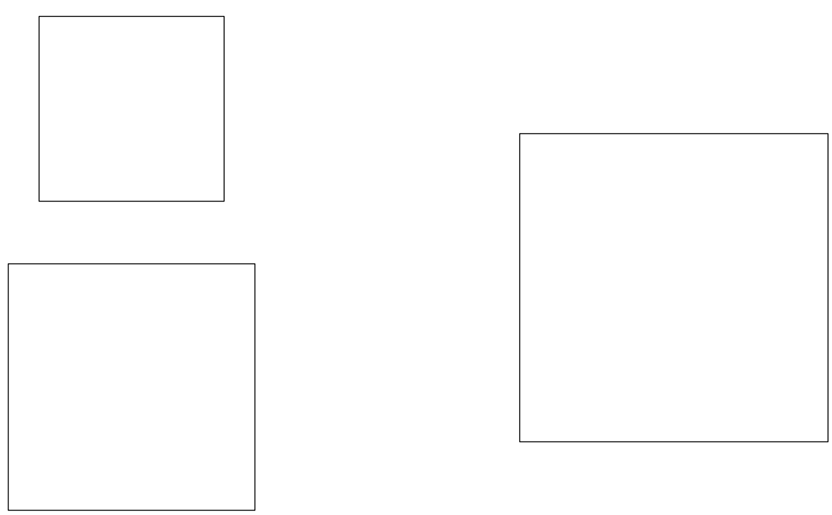
13.



14. The *Pythagorean Theorem* is a classical mathematical theorem about triangles dating back at least to the ancient Greeks. It states that for a triangle having one angle of 90° and side lengths a, b, c with c opposite the 90° angle, the following identity always holds:

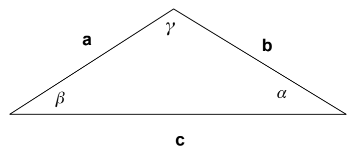
$$a^2 + b^2 = c^2.$$

Prove the Pythagorean Theorem using dissections by dissecting two squares of side length a and b, respectively, into a single square of side length c. You may want to start by trying the squares below. Then, draw your own right triangle, draw square touching all three sides, and try to dissect the two smaller squares into the large one.



15. **Note**: This problem uses the trigonometric function *cosine*. If you have not seen cosine before, that is okay! You can still do this problem just using the geometric explanation at the end of the question.

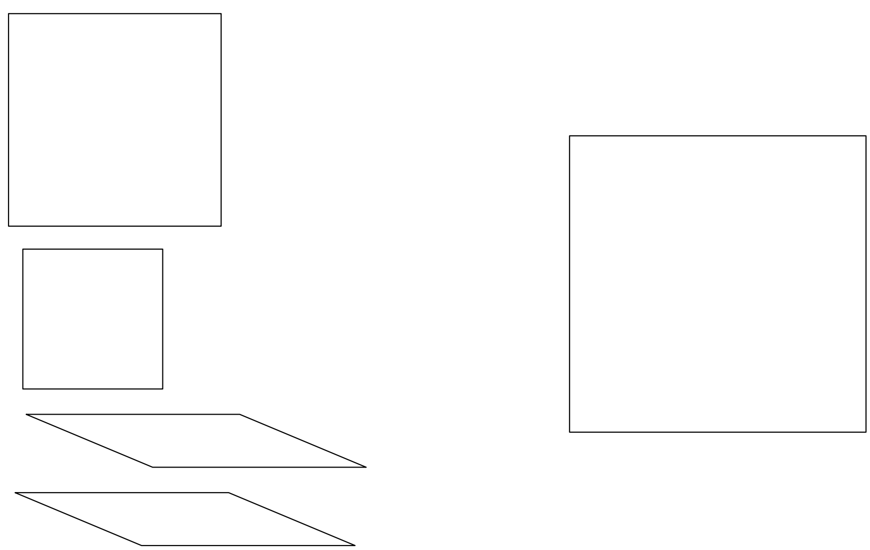
The *Law of Cosines* is a generalization of the Pythagorean Theorem. Given a triangle



with angles α, β, γ and sides a, b, c with a opposite α, b opposite β, and c opposite γ, the following identity always holds:

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma).$$

For $\gamma > 90°, -2ab\cos(\gamma)$ is positive, and a parallelogram having side lengths a and b and having smaller angle $\gamma - 90°$ has area $ab\cos(\gamma)$. Prove the Law of Cosines using dissections by dissecting two squares of side length a, b and two paralellograms each of area $ab\cos(\gamma)$ into a single square.



2 Tools

Now that you've tried your hand at some dissections, we will discuss some tools that can help us.

2.1 Tessellations

A *tessellation*, or *tiling* is a covering of $2D$ space by shapes so that no two overlap. For example, the square grid is a common (and as we will see, very useful!) tessellation.

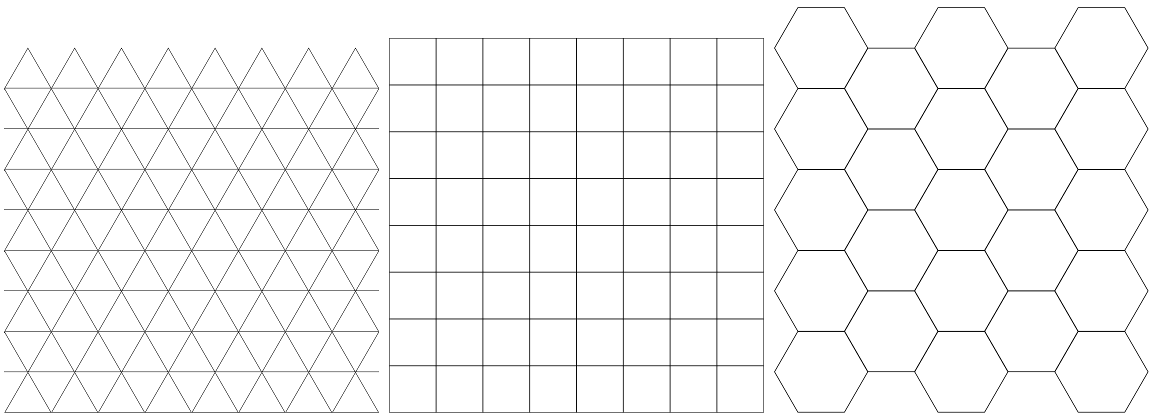
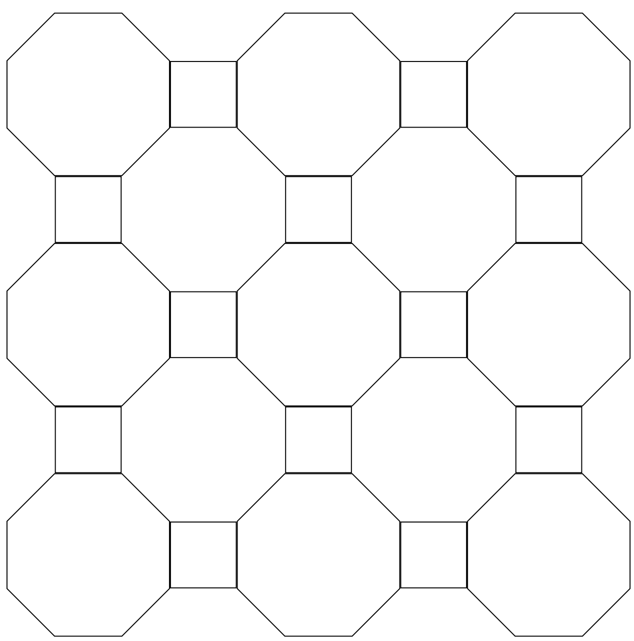


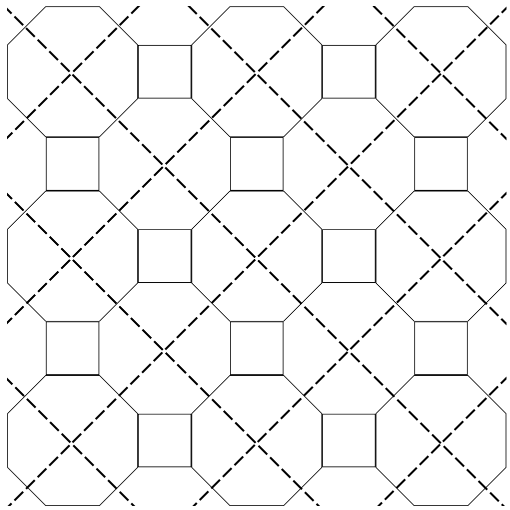
Figure 1: tessellations of the plane with only one regular polygon

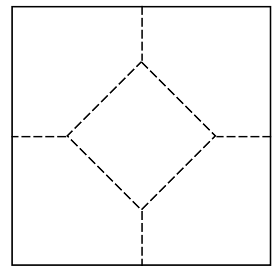
If we superimpose two tessellations, this will tell us how to dissect the shapes of one tessellation to the other.

For example, we can take a tessellation by octagons and squares:

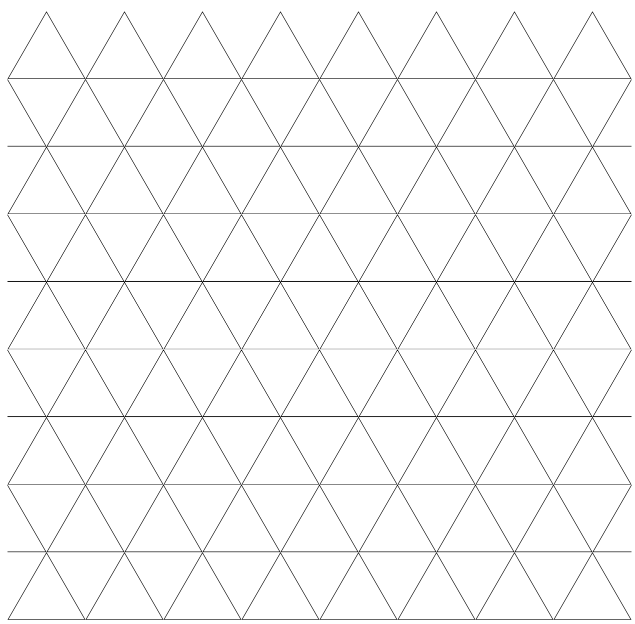


If we carefully overlay the square grid as on the left, we can see how to cut the square into five pieces to make an octagon and square, shown on the right.

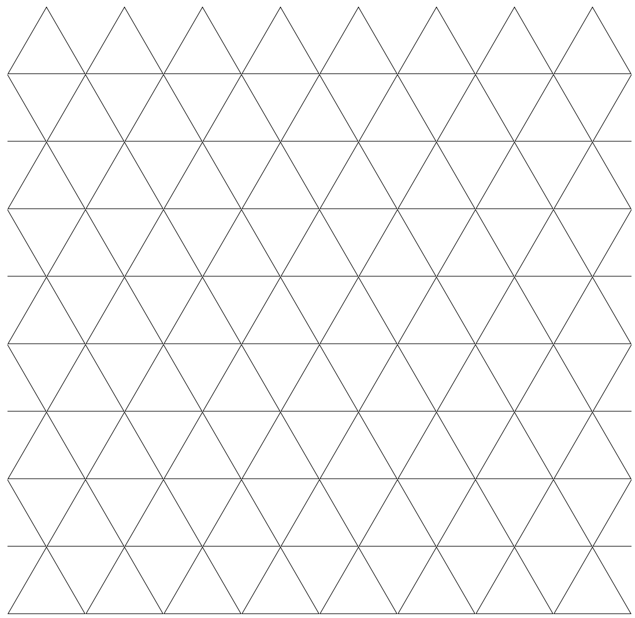




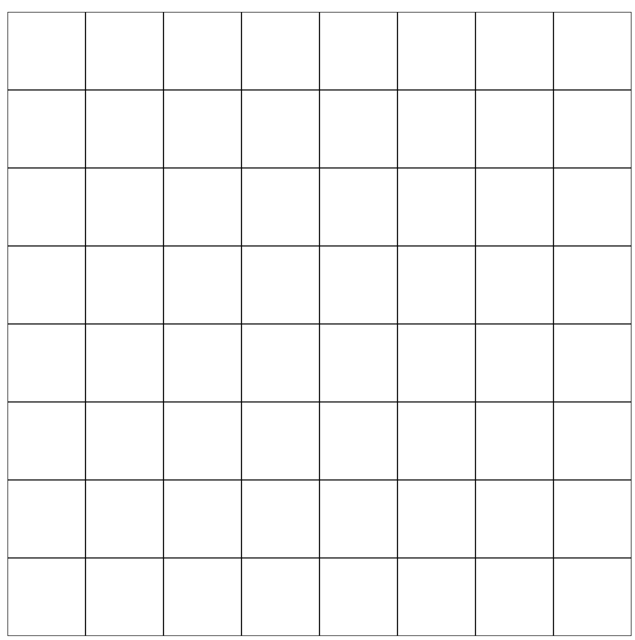
16. Use the triangular tiling to dissect a hexagon into two triangles.



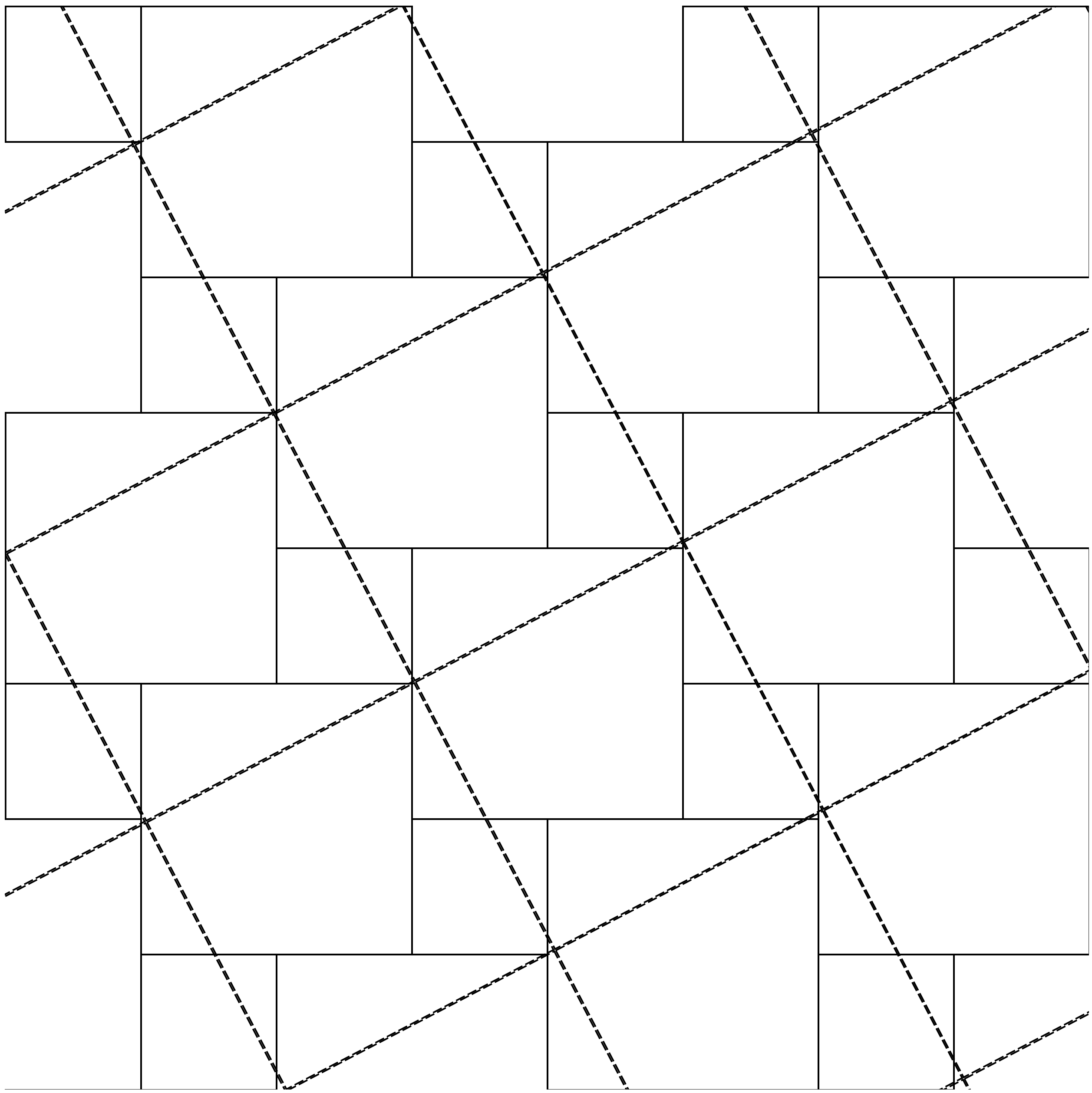
17. Use the triangular tiling to dissect three triangles into one.



18. Use the square grid to dissect two squares of equal size into one.



19. Use the following pair of overlaid tessellations to prove the Pythagorean Theorem:



2.2 P-slides

There is a special construction, called a *P-slide*, that lets you turn one parallelogram into another parallelogram with the same angles but different bases.

If $ABCD$ is a paralellogram and you want a new paralellogram with the same angles but base length shorter than AB, add a point E along AB so that AE has the desired length, and add a point G along CD so that CG is of the desired length. Cut the segment DB, make a cut parallel to DA starting at G, and rearrange as in the diagram.

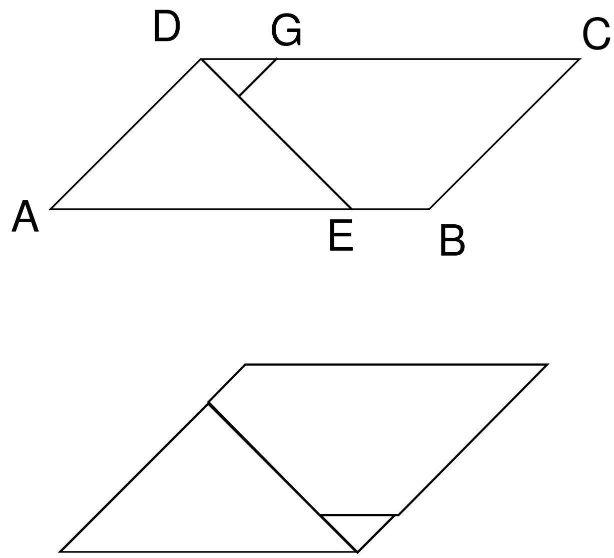
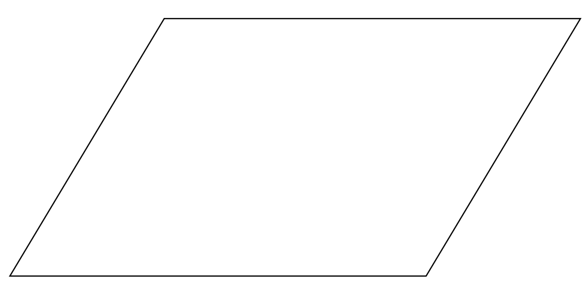
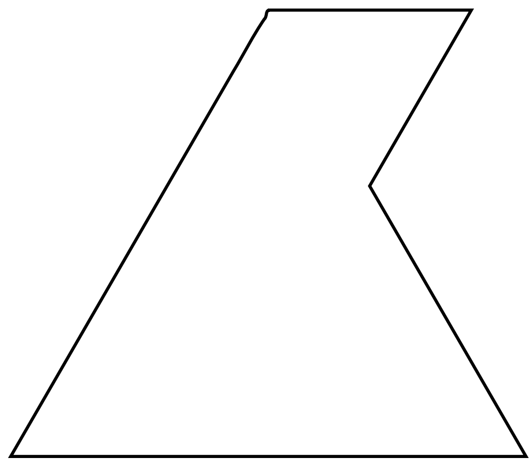


Figure 2: demonstration of a P-slide

20. Try it yourself with the following parallelogram!



21. Use a P-slide to turn the following 5-gon into a trapezoid.



22. Can any two parallelograms with the same angles be dissected to each other by P-slides? Think back to problems 5-6. Can you always do it with one P-slide?

23. Rectangles are a special case of parallelograms, so we can perform P-slides on them! But how many total cuts will we need? Fill out the following table with the fewest cuts you can come up with to turn a $1 \times n$ rectangle into a square. You may want to use grid paper!

n	fewest cuts
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	