Name: ________________________________

UW Math Circle

Week 29 – Infinity

You've probably heard teachers say before, that "infinity is not a number." Maybe that was followed by, "infinity is a concept." But what really is infinity? Today, we'll explore this question.

Here's a first answer: infinity is a size.

- Set A is *at least as big as* set B, denoted $|A| \geq |B|$, if you can pair up the elements of B with distinct elements of A (but there may or may not be elements of A left over).
- Sets A and B *have the same size*, denoted $|A| = |B|$, if it is possible to pair up the elements with nothing left over from either set.
- The fancy letter $\mathbb{N}$ denotes the natural numbers $\{1, 2, 3, 4, \ldots\}$.
- A set has infinite size if it is at least as big as $\mathbb{N}$.

For example, the top and bottom rows here are paired up below, so they have the same size.

Audrey	Brandon	Cadence
|	|	|
soccer	basketball	hockey

On the other hand, the below shows that $\mathbb{N}$ is at least as big as $\{\text{Audrey}, \text{Brandon}, \text{Cadence}\}$.

Audrey	Brandon	Cadence		
|	|	|		
1	2	3	4	$\cdots$

There is no way to pair $\mathbb{N}$ with $\{\text{Audrey}, \text{Brandon}, \text{Cadence}\}$ without having things in $\mathbb{N}$ left over, so $\{\text{Audrey}, \text{Brandon}, \text{Cadence}\}$ is not infinite.

1. Use a pairing to explain why $\{\text{Audrey}, \text{Brandon}, \text{Cadence}\}$ is at least as big as $\{\text{milk}, \text{juice}\}$.

2. Consider all the numbers on the number line, including things like $\sqrt{2}$ and π. This set of numbers is called the real numbers $\mathbb{R}$. Draw a pairing to explain why $\mathbb{R}$ has infinite size.

3. Does $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ have the same size as $\{0, 1, 2, 3, \dots\}$? If so, draw a pairing that proves this. If not, explain why they cannot be paired.

4. Does $\mathbb{N}$ have the same size as the set of positive even numbers? If so, draw a pairing that proves this. If not, explain why they cannot be paired.

5. Does $\mathbb{N}$ have the same size as the set of integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$? If so, draw a pairing that proves this. If not, explain why they cannot be paired.

We say that a set is *countable* if there is a way to list all of the elements (starting from a "first" element, then a second, and so on).

6. Explain why a set A being countable is equivalent to "$\mathbb{N}$ is at least as big as A". Conclude that a set has the same size as $\mathbb{N}$ if it is both countable and infinite.

7. A positive fraction is a number $\frac{a}{b}$, where a and b are natural numbers from $\{1, 2, 3, 4, \dots\}$.
   (a) Find a way to list all of the positive fractions.

   (b) Use the previous part to explain why the set of positive fractions has the same size as $\mathbb{N}$.

   (c) Finally, put this together with ideas from previous questions to explain why the set of all fractions, both positive and negative, still has the same size as $\mathbb{N}$!

8. In math, we usually like to ignore the dictionary when talking about words. That means "word" means any string of letters from the English alphabet, whether or not you can find it in the dictionary. For example, "math" and "ahswmxh" are both words.
   (a) Find a way to list all the words with 1 letter.

   (b) Find a way to list all the words with 2 letters.

(c) Find a way to list all the words with 3 letters.

(d) Find a way to list all words. Conclude that the set of all words has the same size as $\mathbb{N}$.

9. (Challenge) Suppose I had a countable number of countable sets. In other words, I have sets $A_1, A_2, A_3, A_4, \ldots$ and they are all countable, for example, maybe A_1 is the set of all even numbers, A_2 is the set of all English words, and so on. Explain how to list all of the elements in all of the sets, showing that the set containing everything in any of the A_i's is countable.

So far, we've seen something quite surprising: a lot of sets that look like they have more elements than $\mathbb{N}$, actually have the same number of elements as $\mathbb{N}$, because we can pair them up! But are there any sets that cannot be paired up with $\mathbb{N}$ without leaving things left over?

In Problem 2, you showed that the set of numbers on the number line, denoted $\mathbb{R}$, has infinite size. Remember that to be the same size as $\mathbb{N}$, the set needs to be both infinite and countable. Is $\mathbb{R}$ countable? It seems hard — it doesn't feel like there is a natural way to list them all, but how do you prove this?

In 1891, George Cantor published his "diagonalization argument". Let's focus on just the numbers between 0 and 1, and suppose that there actually was a way to list all the real numbers. Then, we will show that no matter how you listed them, you actually missed a number, so it's not possible to list them all. Draw the following chart:

		1	2	3	4	5	6	7	
1	0.	1	7	8	3	0	2	6	$\cdots$
2	0.	2	4	1	5	1	5	5	$\cdots$
3	0.	6	8	9	9	3	0	5	$\cdots$
4	0.	7	9	9	6	6	4	5	$\cdots$
5	0.	8	8	1	3	6	8	8	$\cdots$
6	0.	9	0	9	5	4	7	7	$\cdots$
7	0.	3	6	2	9	2	6	8	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Take the digits along the diagonal, here: $0.1496678\ldots$, and add 1 to every digit, with 9 going to 0. That makes the number $0.2507789\ldots$.

10. Let's finish Cantor's diagonal argument.

    (a) Explain why this new sequence of digits does not belong to the original list.

    (b) We made a mistake! The following chart shows the problem (imagine any digits in the blanks). Explain why if the real numbers were listed like this, the new sequence of digits described above is not a new number.

		1	2	3	4	5	
1	0.	0	9	9	9	9	$\cdots$
2	0.		9				$\cdots$
3	0.			9			$\cdots$
4	0.				9		$\cdots$
5	0.					9	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

(c) Come up with a slightly modified way to make a new number that does not belong to the chart. Use it to explain why the number line is not countable, and thus there are more numbers in $\mathbb{R}$ than in $\mathbb{N}$.

11. The *power set* of a set A, denoted $\mathcal{P}(A)$, is the set of subsets of A. A subset means picking some (possibly none, possibly all) of the elements in A. For example, if A is $\{1,2,3\}$, then $\mathcal{P}(A)$ has 8 elements: $\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$.

    (a) What is $\mathcal{P}(\{8,9\})$?

    (b) Give two examples of elements of $\mathcal{P}(\mathbb{N})$.

    (c) Use a pairing to explain why $\mathcal{P}(A)$ is at least as big as A.

(d) Use a diagonal argument to explain why $\mathcal{P}(\mathbb{N})$ is actually definitely bigger than $\mathbb{N}$. (Hint: explain what are the rows and columns of the following table, and do something similar to Cantor's argument.)

	1	2	3	4	5	6	7	
1	Y	Y	Y	Y	N	Y	Y	$\cdots$
2	N	Y	N	N	Y	Y	N	$\cdots$
3	Y	Y	N	N	N	Y	Y	$\cdots$
4	N	N	Y	Y	Y	N	Y	$\cdots$
5	Y	Y	Y	Y	Y	N	N	$\cdots$
6	Y	N	N	N	N	Y	Y	$\cdots$
7	Y	N	N	Y	N	Y	N	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

12. (Challenge) Explain why $\mathbb{R}$ and $\mathcal{P}(\mathbb{N})$ have the same size. (Hint: yes and no, true and false, 0 and 1?)

Lastly, we'll use our discussion about infinity to learn something about computers and programming! It's okay if you've never done programming before, we'll just talk about programming very generally.

Programmers try to solve problems using computers. A *decision problem* is a special kind of problem where for every input, the desired output is either yes or no. For example, "given a location, compute the fastest route to UW" is not a decision problem, but "given a location, decide whether or not you can drive to UW within 30 minutes" is a decision problem.

13. Explain why the number of decision problems is not countable. (And then certainly, the number of problems overall is also not countable.) (Hint: Problem 11)

A *computer program* is just a list of instructions for the computer to follow. These instructions are usually written using some combination of English characters, spaces, and punctuation symbols.

14. Explain why there is a countable number of computer programs. (Hint: Problem 8)

When the computer tries to run a program that solves a decision problem, it takes in an input, then either gets stuck in an infinite loop or outputs yes/no. We say that a program solves a problem if it never gets stuck in an infinite loop, and the outputs are all correct.

15. Conclude that there is some decision problem that is impossible to compute with any program!

16. Use a diagonal argument to find an explicit function that cannot be computed by any program.