Name: ______________________________

UW Math Circle

Week 28 – Math Olympiad

1 Fun Problems

Today, we're doing some Math Olympiad problems! Your goal in each problem is to answer the question, **prove that your answer is correct**, and present it to your group.

Presentation guidelines:

1. **Start by** stating the problem in your own words, and say what you're going to prove.
2. Summarize the main steps of your proof! *(Eg. if you're doing the proof in two cases, explain what the cases are.)*
3. Present your proof, and **explain every step in detail.** Take your time. Mathematicians are skeptical, so make sure there's no room for doubt!
4. **If something's important, write it down!** This includes any equations or pictures involved in your proof. Your written work should make sense on its own. If there's a picture, label it clearly. If there are equations, write down what each variable represents.

1. (UWMO 2010) Is it possible to draw some number of diagonals in a convex hexagon so that every diagonal crosses EXACTLY three others in the interior of the hexagon? (Diagonals that touch at one of the corners of the hexagon DO NOT count as crossing.)

2. (UWMO 2010) A 3×3 square grid is filled with positive numbers so that

   (a) the product of the numbers in every row is 1,

   (b) the product of the numbers in every column is 1,

   (c) the product of the numbers in any of the four 2×2 squares is 2.

   What is the middle number in the grid? Find all possible answers and show that there are no others.

3. (UWMO 2010) Each letter in ELROND's name represents a distinct digit between 0 and 9. Show that

$$ELROND \times E \times L \times R \times O \times N \times D$$

is divisible by 3. (For example, if $E = 1, L = 2, R = 3, O = 4, N = 5, D = 6$, then $ELROND \times E \times L \times R \times O \times N \times D = 123456 \times 1 \times 2 \times 3 \times 4 \times 5 \times 6$).

4. (UWMO 2010) You walk into a room and find five boxes sitting on a table. Each box contains some number of coins, and you can see how many coins are in each box. In the corner of the room, there is a large pile of coins. You can take two coins at a time from the pile and place them in different boxes. If you can add coins to boxes in this way as many times as you like, can you guarantee that each box on the table will eventually contain the same number of coins?

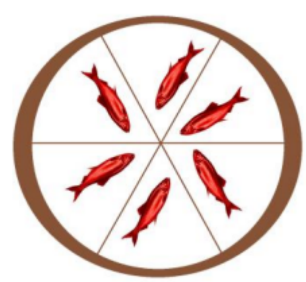
5. (UWMO 2010) Alex, Blake, and Casey are playing a table tennis tournament. During each game, two players are playing each other and one is resting. In the next game the player who lost a game goes to rest, and the player who was resting plays the winner. By the end of tournament, Alex played a total of 10 games, Blake played 15 games, and Casey played 17 games. Who lost the second game?

2 More Fun Problems

1. (UWMO 2011) In a chemical lab there are three vials: one that can hold 1 oz of fluid, another that can hold 2 oz, and a third that can hold 3 oz. The first is filled with grape juice, the second with sulfuric acid, and the third with water. There are also 3 empty vials in the cupboard, also of sizes 1 oz, 2 oz, and 3 oz. In order to save the world with grape-flavored acid, James Bond must make three full bottles, one of each size, filled with a mixture of all three liquids so that each bottle has the same ratio of juice to acid to water. How can he do this, considering he was silly enough not to bring any equipment?

2. (UWMO 2011) Twelve people, some are knights and some are knaves, are sitting around a table. Knaves always lie and knights always tell the truth. At some point they start up a conversation. The first person says, "There are no knights around this table." The second says, "There is at most one knight at this table." The third – "There are at most two knights at the table." And so on until the 12th says, "There are at most eleven knights at the table." How many knights are at the table? Justify your answer.

3. (UWMO 2011) Aquaman has a barrel divided up into six sections, and he has placed a red herring in each. Aquaman can command any fish of his choice to either 'jump counterclockwise to the next sector' or 'jump clockwise to the next sector.' Using a sequence of exactly 30 of these commands, can he relocate all the red herrings to one sector? If yes, show how. If no, explain why not.



4. (UWMO 2011) Is it possible to place 13 integers around a circle so that the sum of any 3 adjacent numbers is exactly 13?

5. (UWMO 2011) Two gamers are playing a game. The first player writes the letters A or B in a row, left to right, adding one letter on their turn. The second player switches any two letters after each move by the first player (the letters do not have to be adjacent), or does nothing, which also counts as a move. The game is over when each player has made 2011 moves. Can the second player plan their move so that the resulting letters form a palindrome? (A palindrome is a sequence that reads the same forward and backwards, e.g. AABABAA.)