Name: ____________________________

UW Math Circle

Week 27 – Parametric Equations

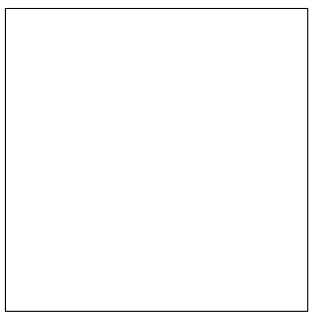
1 Drawing with Math

Imagine there is a robot holding a pen. The robots arm can move **forward**, **backward**, **left**, and **right**, and only follows very specific, precise instructions.
Come up with instructions for the robot to draw the following shapes!

1. How would you tell the robot to draw this square?

4in

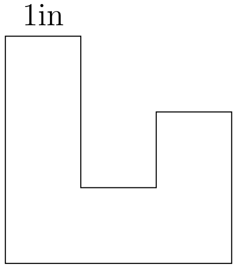
4in



2. What about this shape?

1in

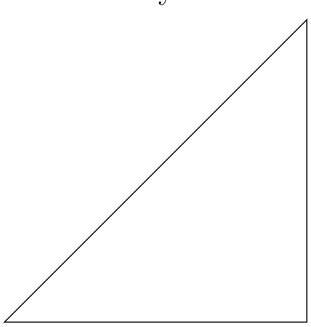
3in



2in

3in

3. How could you instruct the robot to draw this triangle?



4in

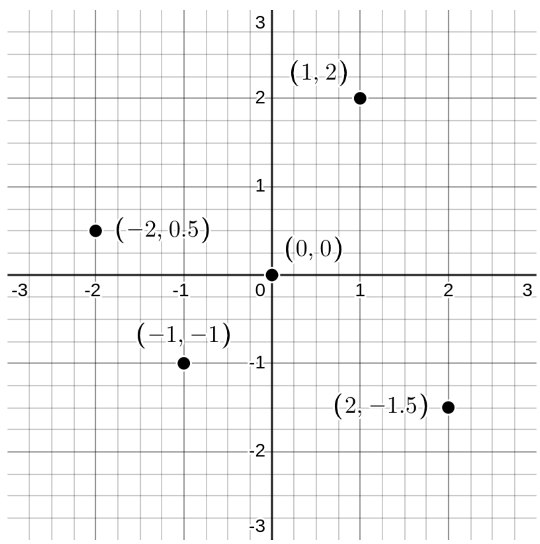
4in

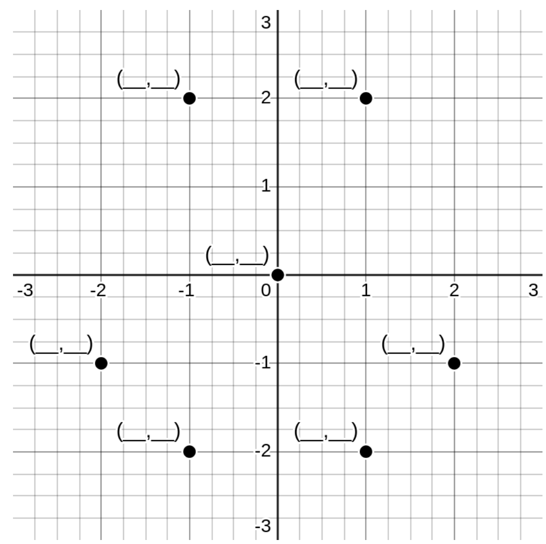
2 Cartesian Plane

In order to draw more shapes, our robot will take instructions by viewing the paper as a *cartesian plane*. This is a way to describe the position of the pen by measuring where it is horizontally and vertically on the paper.

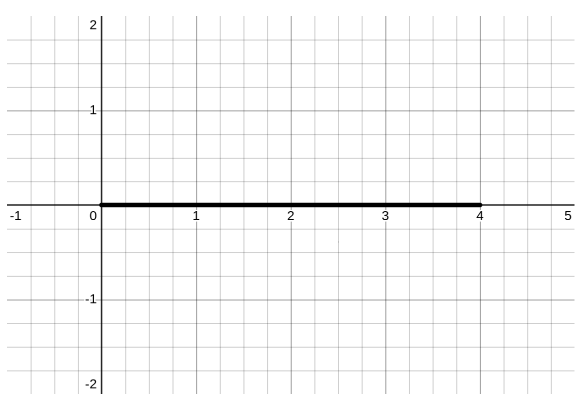
We label our page as a grid and choose one point to be our center, called the *origin*. We can label a point that is x inches to the right of the origin and y inches above the origin as (x, y). For example, we have labeled the points on the plane on the left.

4. Label the points on the cartesian plane on the right.





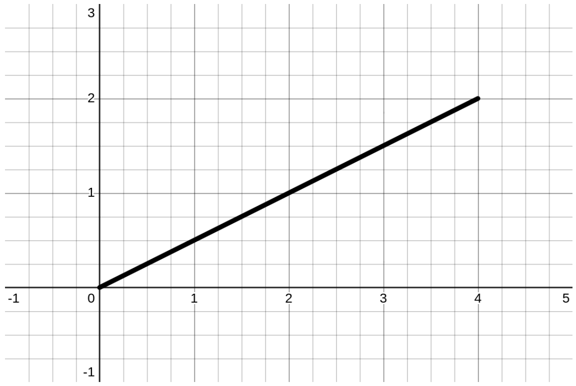
5. Now let's use this method to give explicit instructions to our robot. We want to draw a horizontal four inch line in four seconds. We start at the origin $(0, 0)$, and four seconds later the pen is at $(4, 0)$.



(a) If the pen is moving at a constant speed, at what coordinate is the pen after 2 seconds? One second?

(b) Where is the pen after t seconds? Write the coordinate in terms of t.

6. Let's draw a more interesting line. We want to start at the origin and draw a straight line to the point $(4, 2)$ in four seconds.

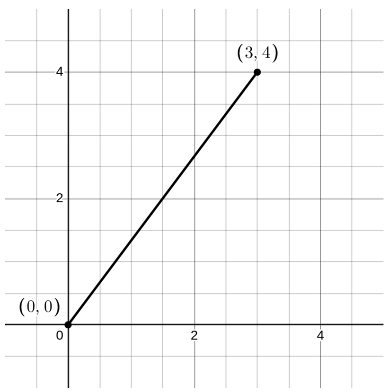


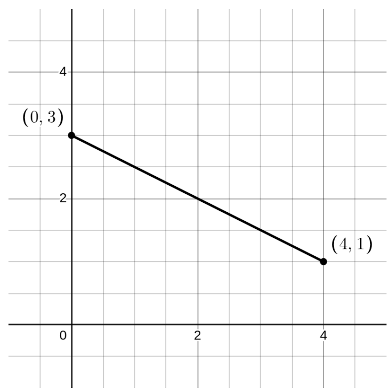
(a) If the pen is moving at a constant speed, at what coordinate is the pen after 2 seconds? One second?

(b) Where is the pen after t seconds? Write the coordinate in terms of t.

The equation describing the x- and y-coordinate in terms of time t is called a **parametric equation**. The image a parametric equation produces is called a **curve**.

7. Write parametric equations that draw each of the following lines. Remember to write the time interval so the robot knows when to start and when to stop drawing!



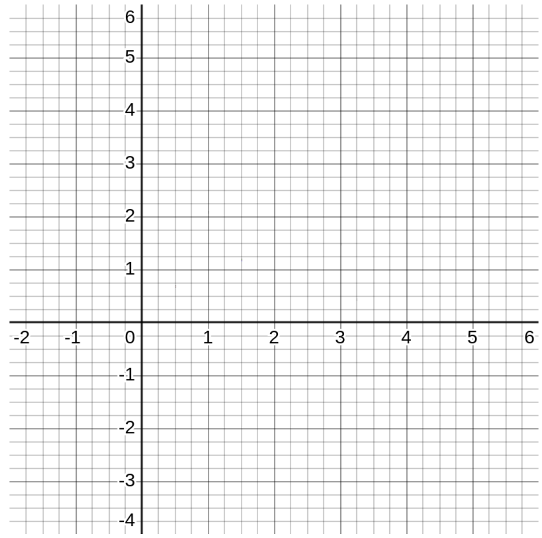


8. Can two different parametric equations draw the same curve? Give an example or explain why not.

9. Draw the curve defined by the parametric equation $(x(t), y(t))$ defined below on the interval $0 \leq t \leq 4$. Fill out the table of coordinates for each time step and graph the point to help draw the curve.

$$x(t) = (t-2)^2$$
$$y(t) = 2t - 3$$

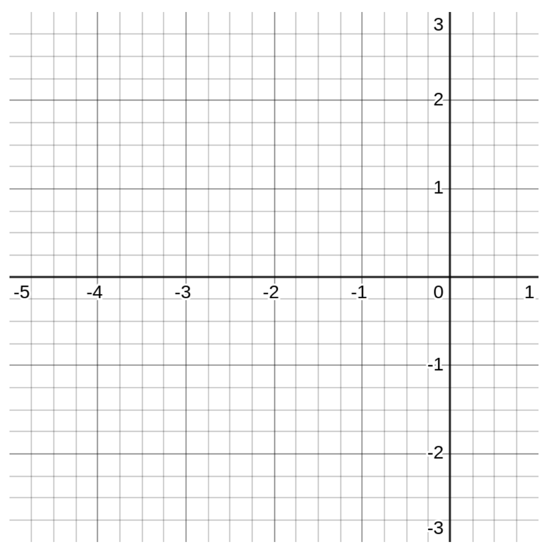
t	x	y
0		
1		
2		
3		
4		



10. Draw the curve defined by the parametric equation $(x(t), y(t))$ on the interval $-2 \leq t \leq 2$, where

$$x(t) = t^2 - 4$$
$$y(t) = -\frac{1}{4}t^3 + t$$

t	x	y
-2		
-1		
0		
1		
2		



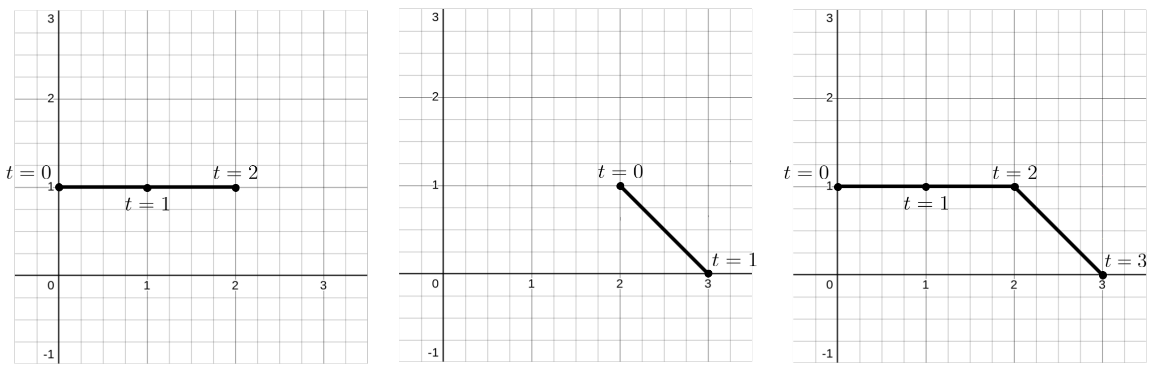
3 Concatenating Curves

If you have two curves and one starts where the other one ends, we can *concatenate* these curves together by telling the robot to draw one, and then immediately draw the other after.

Parametric equations for this new curve can be written using the equations for the two original curves put together "piece-wise." For example, if we wanted to draw the two lines $(t,1)$ for $0 \le t \le 2$ and $(t+2, -t+1)$ for $0 \le t \le 1$, we can use the parametric equations

$$\begin{cases} (t,1) & 0 \le t < 2 \\ ((t-2)+2, -(t-2)+1) & 2 \le t \le 3 \end{cases}$$

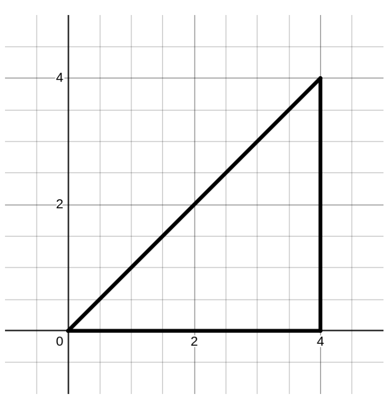
Notice how in the second equation we replaced t with $t-2$. This is so that we don't start drawing the second curve until $t=2$, and then we start in the same position we would have before.



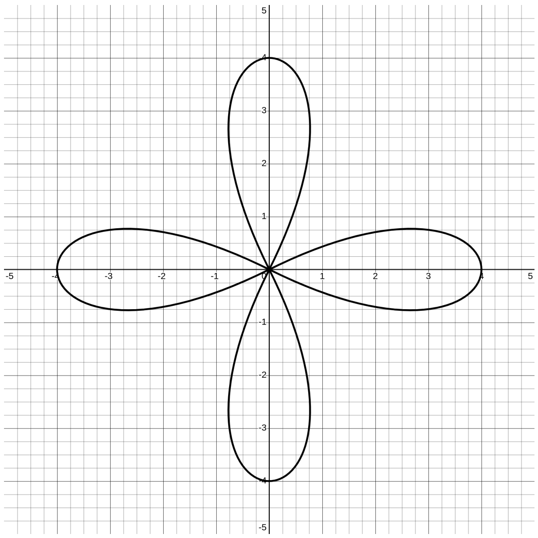
Given $(x_1(t), y_1(t))$ on the time interval $0 \le t \le 1$ and $(x_2(t), y_2(t))$ on $0 \le t \le 1$, their concatenation has parametric equation $(x(t), y(t))$ on the interval $0 \le t \le 2$ where

$$\begin{cases} (x_1(t), y_1(t)) & \text{when } 0 \le t < 1 \\ (x_2(t-1), y_2(t-1)) & \text{when } 1 \le t \le 2 \end{cases}$$

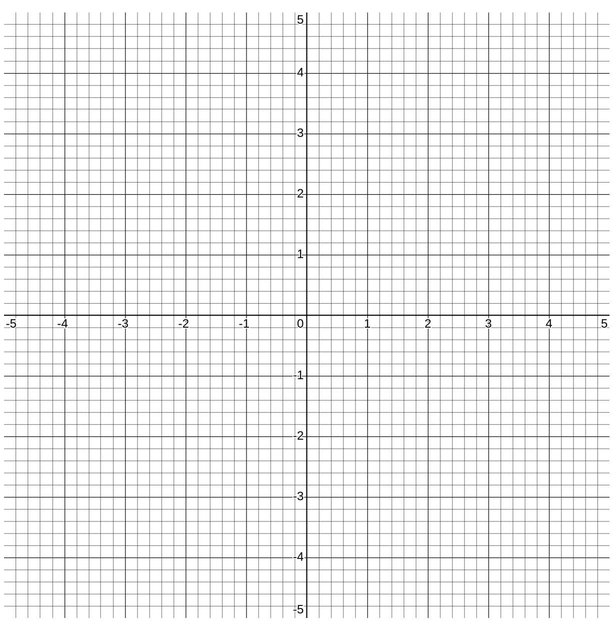
11. Write parametric equations for the triangle in problem 3, graphed below.



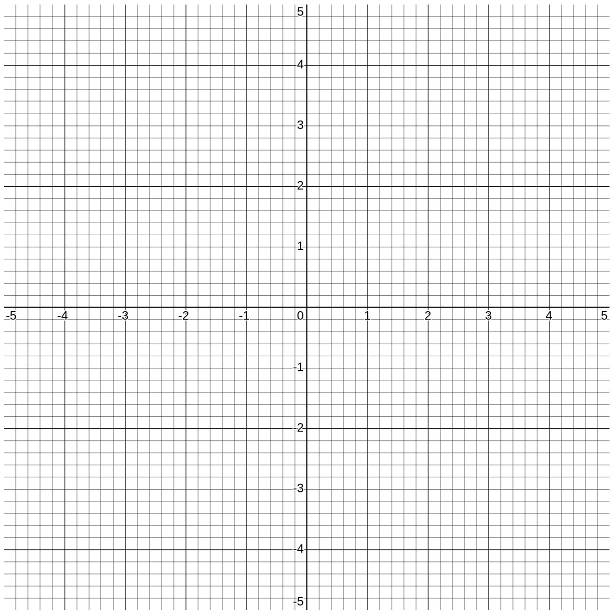
12. The curve from problem 10 should look like the left petal in the flower below. Use the given parametric equations to come up with equations for the remaining three petals, and concatenate them together to get parametric equations for the whole flower.

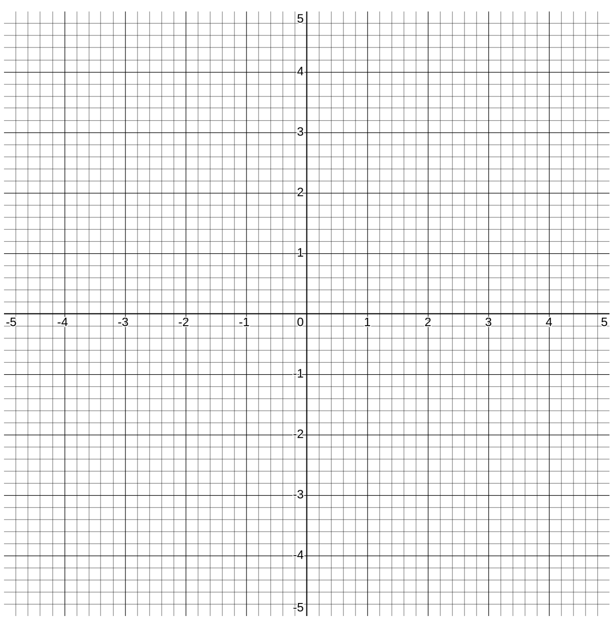


13. Create your own parametric equation that draws a heart.



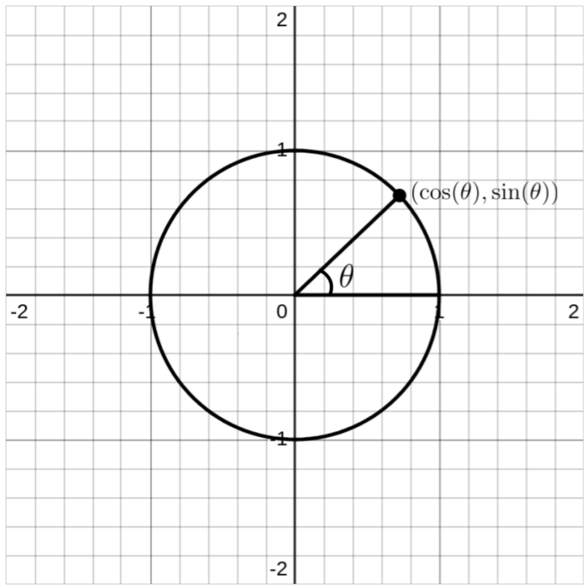
14. Create your own parametric equations that look like any design you want!





4 Circles

Circles are more challenging to draw with cartesian coordinates than lines. One way to parametrize the circle is using *trigonometric functions*. Consider a circle of radius 1. For an angle θ from the positive x-axis, we define $\cos(\theta)$ to be the x-coordinate and $\sin(\theta)$ the y-coordinate of the point on the circle at this angle



Using this, we get a parametric equation for the circle of radius 1 by $(\sin(t), \cos(t))$ for $0 \leq t \leq 360$.

15. Use this definition of sine and cosine to find the following:

    (a) $\sin(0°) =$

    (b) $\sin(90°) =$

    (c) $\cos(180°) =$

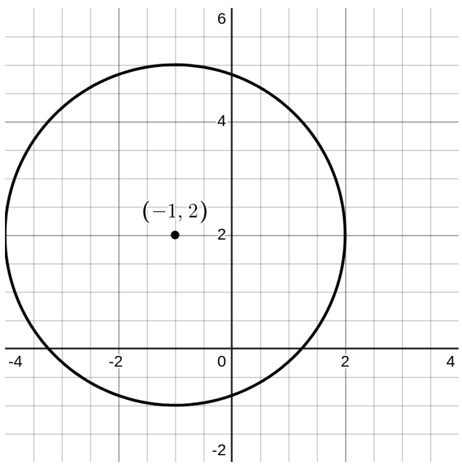
    (d) For what θ does $\sin(\theta) = -1$?

    (e) Find all angles $0° \leq \theta \leq 360°$ where $\cos(\theta) = 0$.

    (f) At what angles does $\sin(\theta) = \cos(\theta)$?

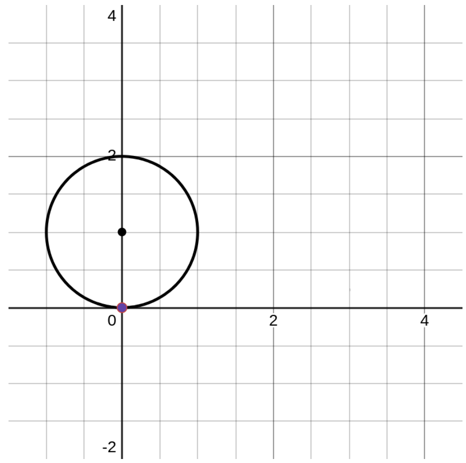
16. Give a parametric equation for a circle of radius 2.

17. Give a parametric equation for a circle of radius 3 centered at the point $(-1, 2)$.



If you want the full circle to be traced out in exactly 4 seconds, what do you need to change?

18. A wheel of radius 1 is rolling on the ground at one rotation per second.

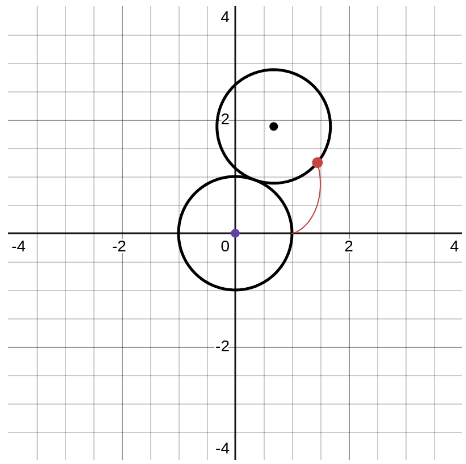


(a) Assuming the bottom of the wheel starts at the origin, give a parametric equation for the position of the center of the wheel between 0 and 4 seconds.

(b) A piece of gum is on the sidewalk at the origin and sticks to the wheel in that spot. Write parametric equations for the position of the piece of gum between 0 and four seconds.

(Hint: Think about the position of the gum in relation to the center of the wheel first! You can test this against an instructor's visual. Keep tweaking your equations until yours works.)

19. Let's try something similar, except this time the wheel is rolling on another wheel fixed at the origin, both of radius one.



(a) The wheel starts on the x axis so it is centered at $(2, 0)$. It takes the wheel four seconds to return to its starting position. Write a parametric equation for the position of the center of the wheel.

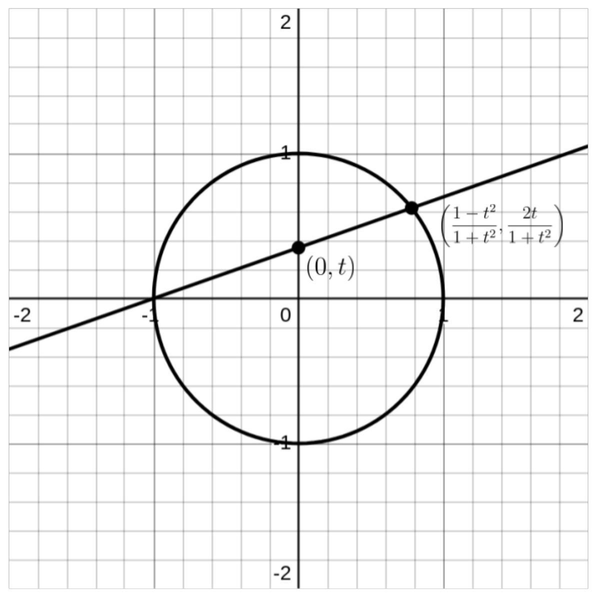
(b) Once again, A piece of gum gets stuck on the moving wheel where the two first touch at $(1, 0)$. Write parametric equations for the position of the piece of gum. This curve is called a *cardioid*.

(Hint: Although the outer wheel only makes one full rotation around the inner wheel, the gum will rotate around the center of the wheel twice. Think about why this is the case.)

You may remember two weeks ago we found a parametrization for the circle by the parametric equation

$$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

This comes from taking the intersection of the line passing through the points $(-1,0)$ and $(0,t)$ with the circle.



Have an instructor graph this for you.

20. Does this parametric equation draw the whole circle? What time range is needed to do this?

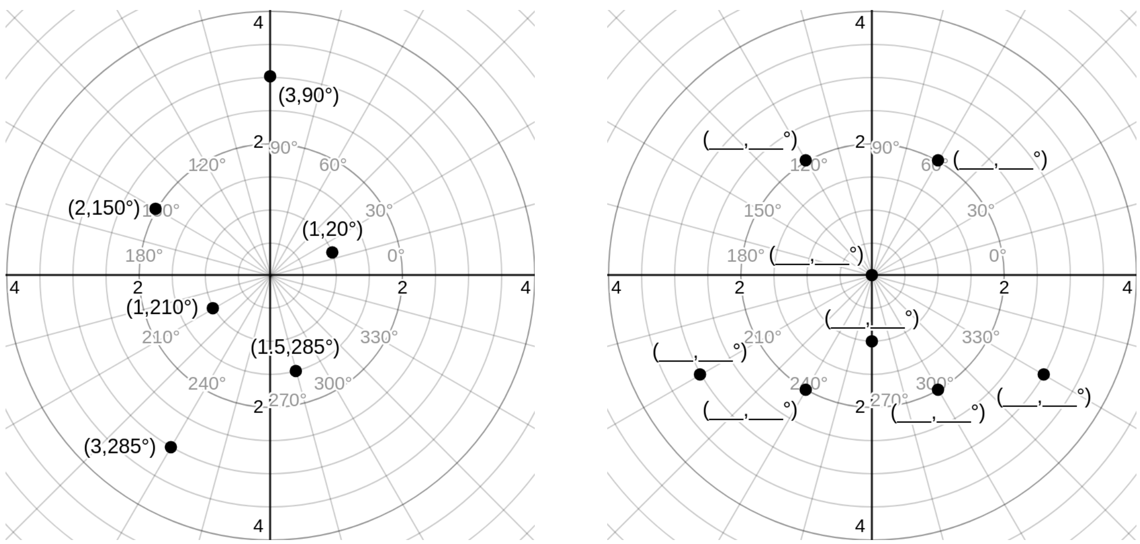
21. Look at the visualization of this parametric equation verses the previous one. In what ways do they differ?

5 Alternative Coordinates

On the cartesian plane, drawing lines is very simple, but drawing curves like circles is a lot more involved. If we instead look at polar coordinates, drawing circles becomes very easy!

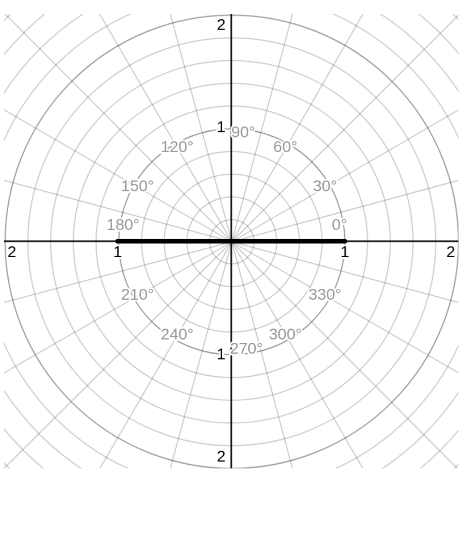
Polar coordinates are similar to cartesian coordinates, except instead of measuring horizontal and vertical distance from the origin, the first coordinate r gives the distance in inches from the origin, and the second coordinate θ gives the angle from the positive x-axis. For example, we have labeled the points on the plane on the left.

22. Label the point on the right using polar coordinates.

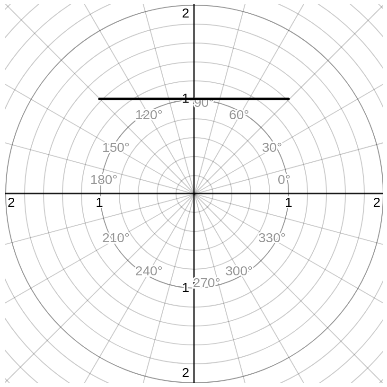


23. Write parametric equations in polar coordinates for a circle of radius 3.

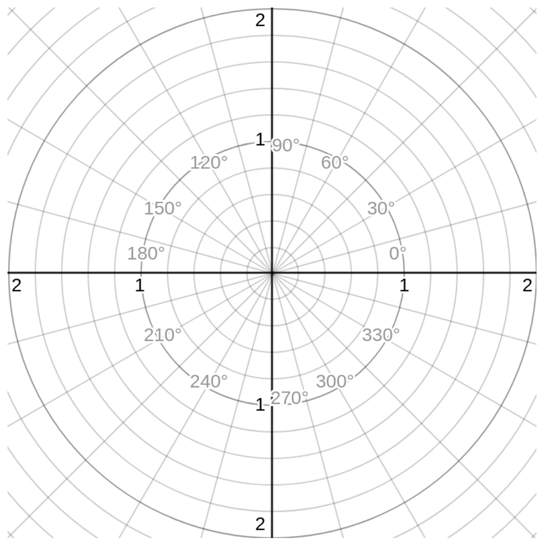
24. Write parametric equations in polar coordinates for the line drawn below.



25. Challenge: Write parametric equations for this line:



26. Graph the curve defined by the polar coordinates $r(t) = t$, $\theta(t) = 360t$ on $0 \leq t \leq 2$.



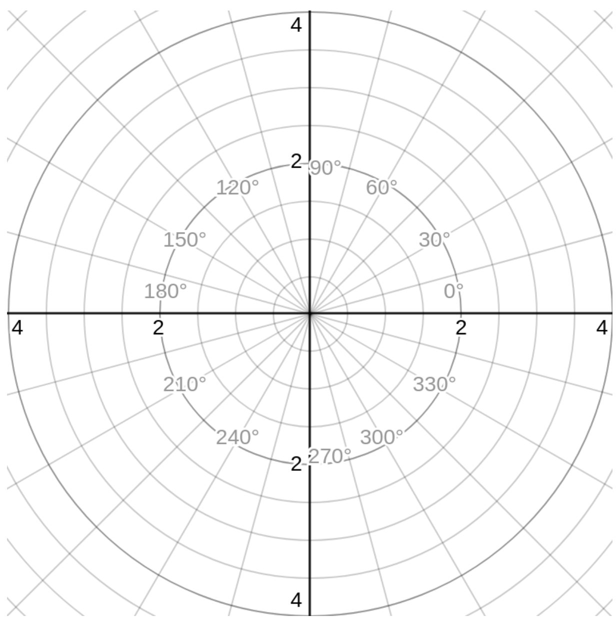
27. For an integer n, the parametric equation $(\sin(nt), t)$ on $0 \leq t \leq 360$ looks like a flower.

(a) Graph some of these and make a conjecture about how many petals these flowers will have. Can you justify this?

(b) Now try with $n = \frac{3}{2}$. You may have to extend the time interval to get the full flower. What happens? Try with some other fractions and make some conjectures.

28. Find parametric equations for the cardioid curve defined in problem 20 part b. What happens if the two circles have different radii? Ask your instructor to show the graph. These curves are called limaçon.

29. Write your own parametric equations in polar coordinates to design anything you like!



30. Invent your own coordinates. Describe how they work, and what curves are easy to parametrize in your coordinates.