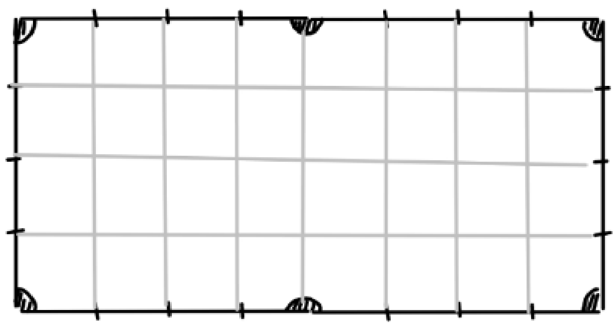
Name: ______________________________

UW Math Circle

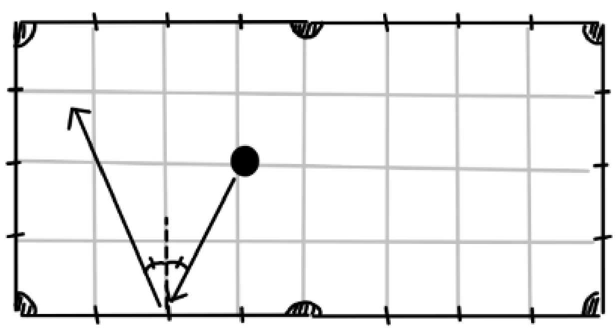
Week 26 – Billiards

1 Billiards

Billiards is a game similar to pool and a Billiards table is drawn below.



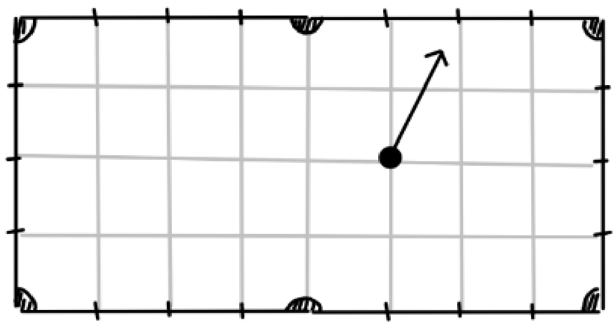
When a player hits a ball, it will continue straight until it hits a wall. When the ball hits a wall, it will reflect off the wall at the same angle it hit the wall (see below).



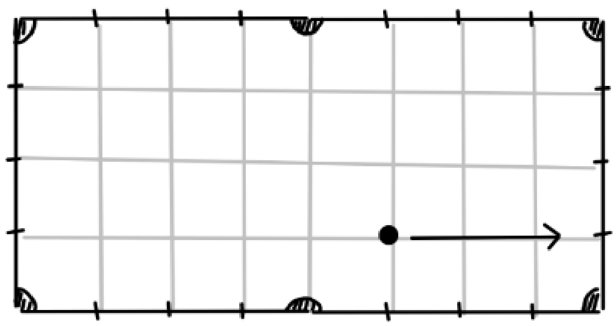
In fact, in the above picture, the ball will bounce off the wall and land in the top left pocket! If the ball does not hit a pocket, we will assume the ball will continue forever at the same speed (like a very good air hockey table).

1. For each of the billiards tables below, predict the trajectory of the ball. Which balls will land in a pocket?

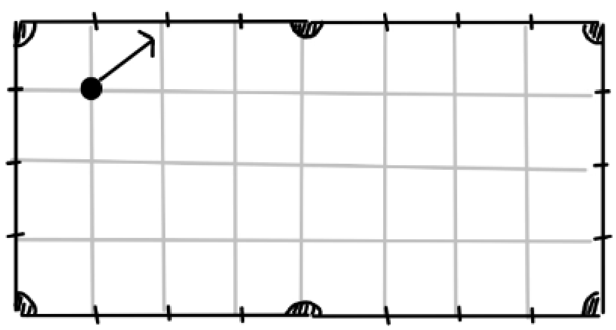
(a)



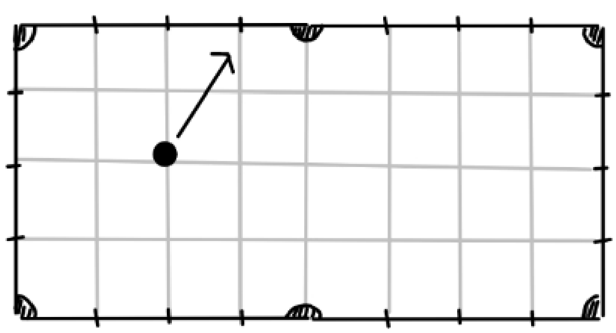
(b)



(c)

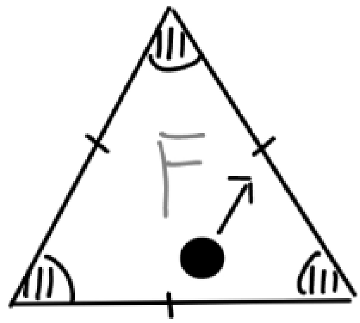


(d)

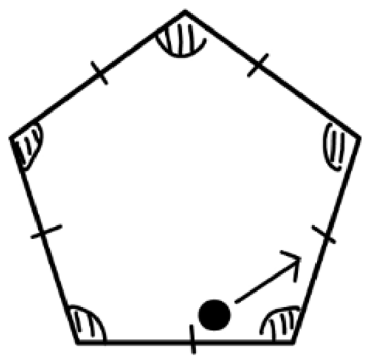


2. Next, we predict the trajectories on different table shapes. For these new table shapes, there are pockets at every corner, but nowhere else.

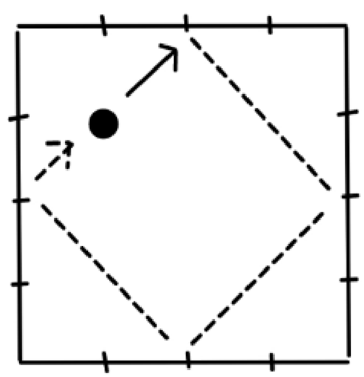
   (a)



   (b)

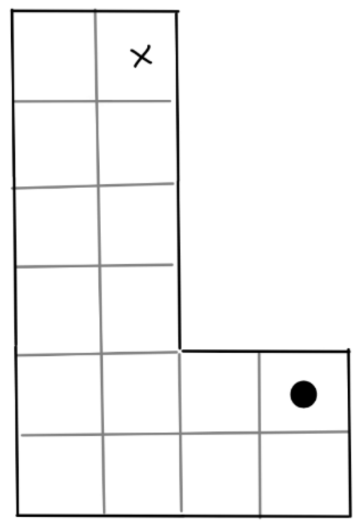


3. Consider the following trajectory on a square table.

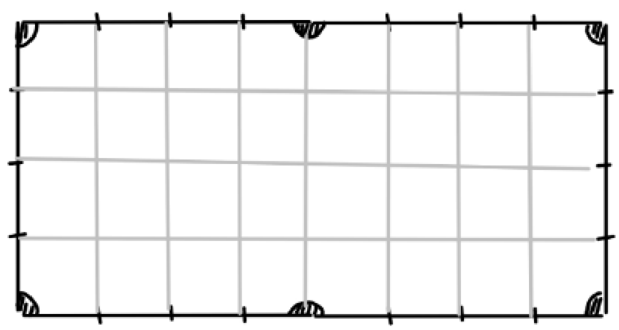


   This trajectory is called *periodic* because the ball returns to its starting point going in the original direction, so the ball will repeat the same trajectory forever. Which of the trajectories in Problem 1 and Problem 2 are periodic?

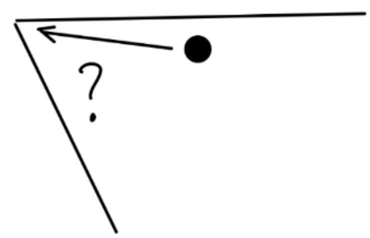
4. How can you hit the following ball so that it goes through the “X” on the table? Can you do it with 3 wall collisions? What about 2? Can you do it with only one?



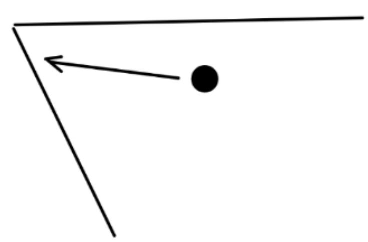
5. To hone your billiards skills, you try to play with a blindfold on. While blindfolded, you place a billiards ball somewhere on the table below, and hit it roughly up and to the right very hard. You hear that the ball makes 2025 collisions with the table before it goes into a pocket. Which pockets could the ball have gone into?

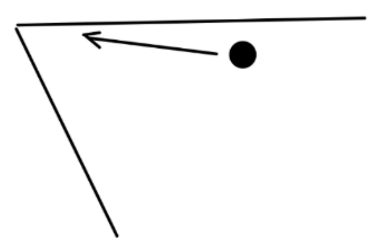


6. From now on, we will think of our billiards ball as a single point. If we remove the pockets from our billiards table, what do you think should happen if the ball hits the corner exactly?



7. Consider the following two initial trajectories. The ball begins in the same direction as in Problem 6, but its initial position is moved slightly so the ball will not hit the corner directly.



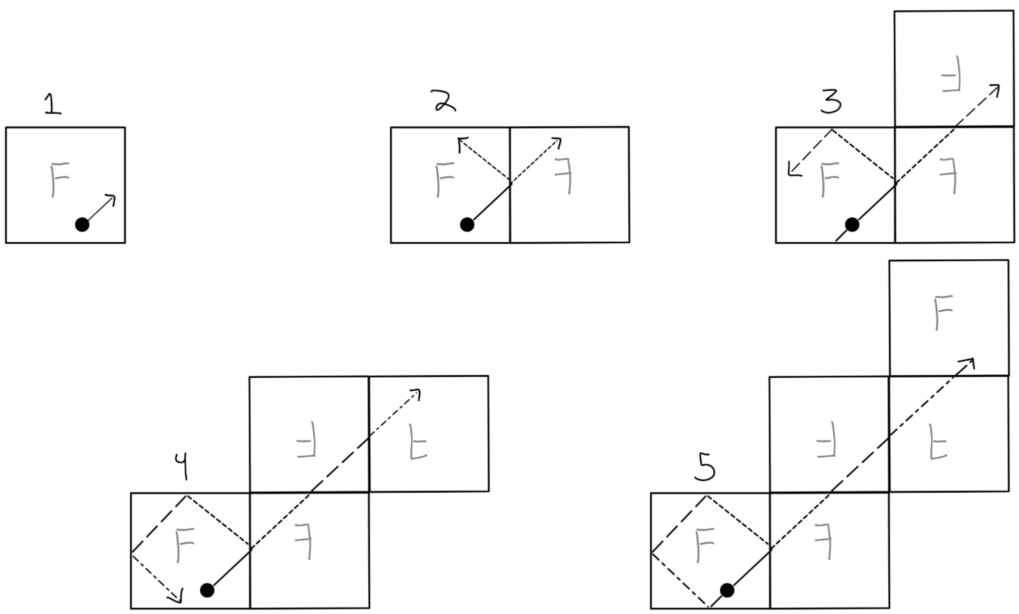


Will the two balls above leave the corner moving in the same direction? Why or why not? Does this change your answer to Problem 6?

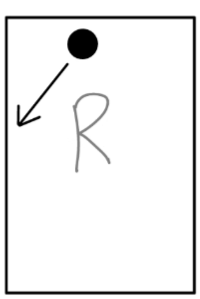
8. (Open Problem) Is there a periodic trajectory on every triangular billiards table?

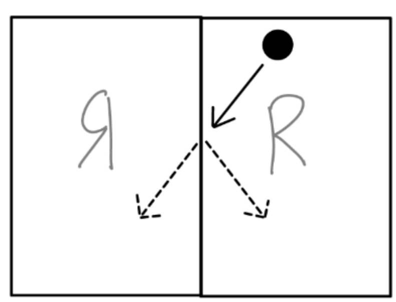
2 The Unfolding Method

For all of the billiard tables that we studied in Section 1, we can use a special trick to draw long trajectories. When the trajectory hits an edge, we draw a mirrored version of the table on the opposite side and continue the trajectory in a straight line. When it hits another edge, add another table and keep going!
By "folding" the diagram back up, we can see how the trajectory looks on the original table!

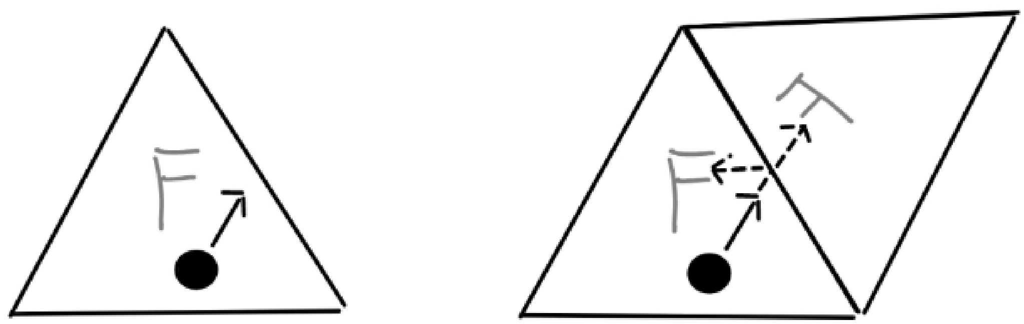


9. Follow the unfolding procedure described at the beginning of this section to the following table and trajectory. The first step has already been done. Continue unfolding until you get an "F" in the original orientation.



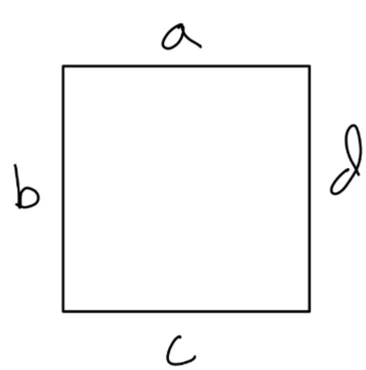


10. Complete the unfolding procedure for the below table until you return to an "F" in the original orientation.



11. You're playing billiards on a square table with sides a, b, c, and d, as shown. You hit the ball and record the order of sides it bounces off. Which of these orders are possible? *(Hint: use the unfolding method!)*

   (i) a c b d
   (ii) a c d a b
   (iii) a b c d a c

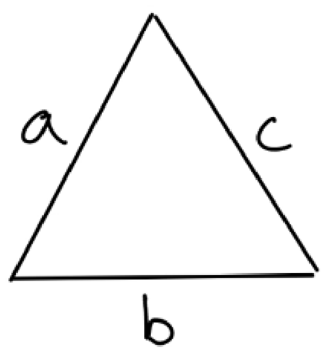


12. Now your table is an equilateral triangle with sides a, b, and c, as shown. Which of these bounce orders are possible?

    (i) a b a b

    (ii) a b c a c

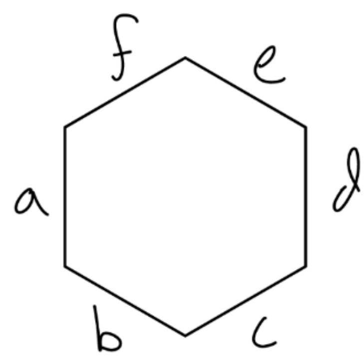
    (iii) b c a b a c b



13. This time your table is a regular hexagon with sides a, b, c, d, e, and f, as shown. Which of these bounce orders are possible?

    (i) a d e

    (ii) a e c b



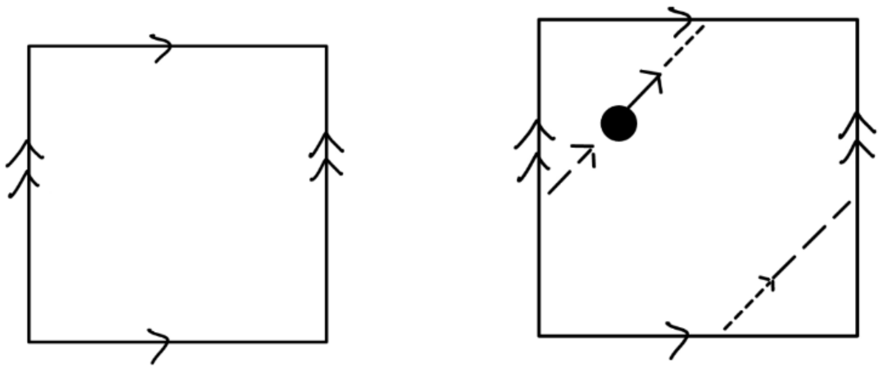
14. You're playing on a 1m × 1m square billiard table with pockets in the four corners. You hit the ball along a line ℓ from the bottom left corner.

    Claim: You'll sink the ball if and only if the slope of ℓ is *rational*.

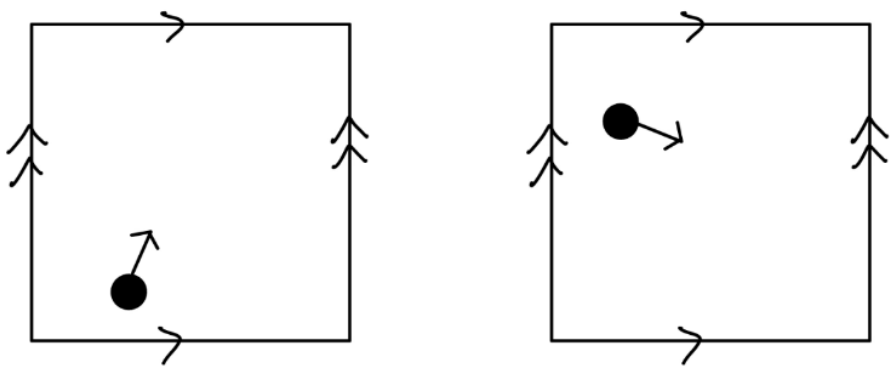
    Use the unfolding method to explain why this claim is true!

3 Translation Surfaces

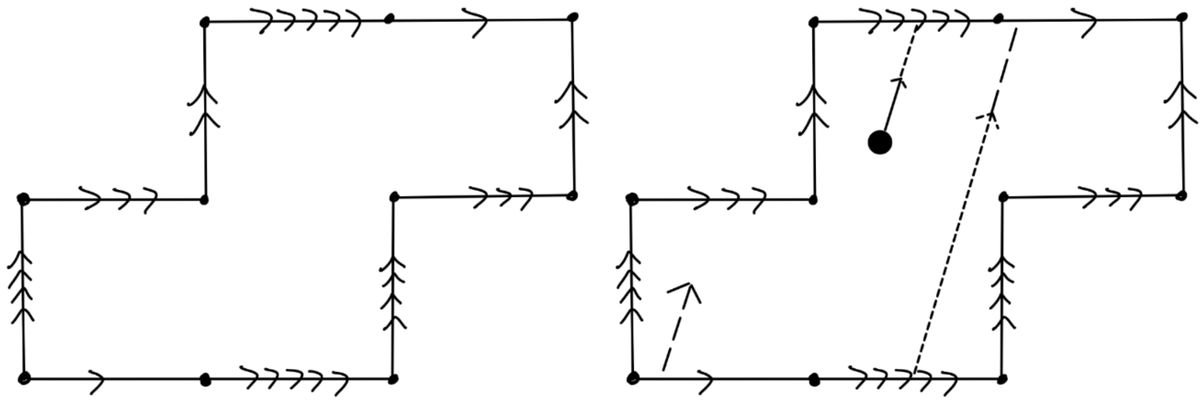
We can glue together the parallel edges of a square, which we indicate by the chevron marks below. Then instead of bouncing off the edges, the billiard ball continues in the same direction from the glued edge (similar to the games Asteroids and Pacman). This billiards table is called the *torus*. Throughout this section we will see other tables created by gluing together parallel edges, and such tables are called *translation surfaces*.



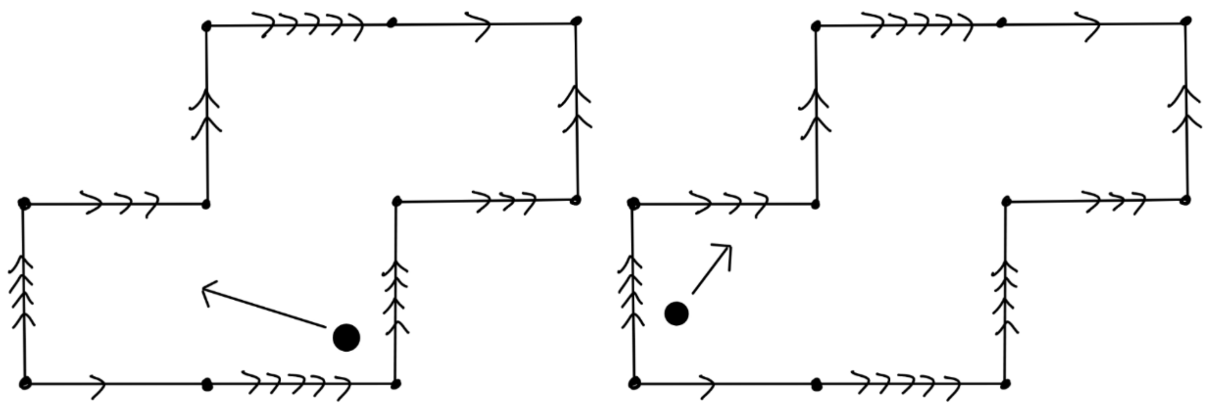
15. Predict the approximate trajectory of the following ball on the torus.



16. We can create more tables similar to the torus by gluing together parallel edges of the same length on a polygon. For example, consider the following table where the edges with one chevron are glued together, the edges with two chevrons are glued together, etc.

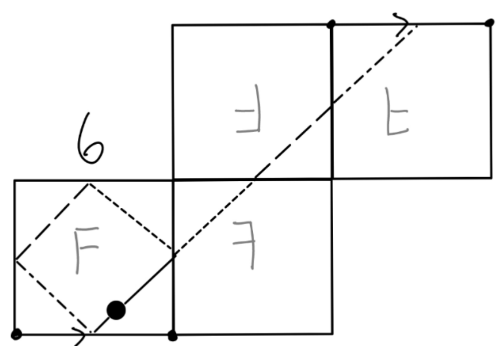


Approximately predict the following trajectories on this table.



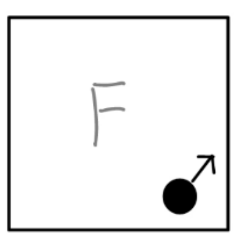
4 Unfolding and Translation Surfaces

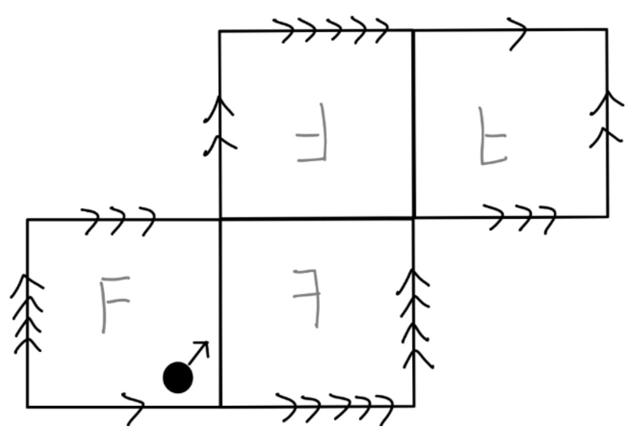
Return to the description of unfolding at the beginning of Section 2. Note that after a few reflections we end up with the table in the top right that is in the same orientation as our original table in the bottom left (i.e. when the "F" faces the right way). In fact, instead of drawing this fifth table in the top right, we can glue the edges so that the trajectory returns to the original table.

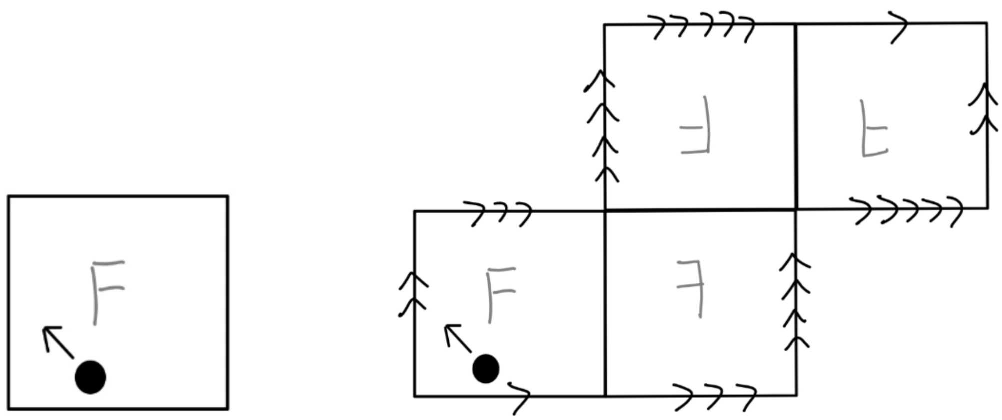


17. We can glue together a few more edges from the above construction to create the following translation surfaces. Draw the following trajectories as both a billiards ball bounding off the walls and as a path on the translation surface. Do these paths agree? Why or why not?

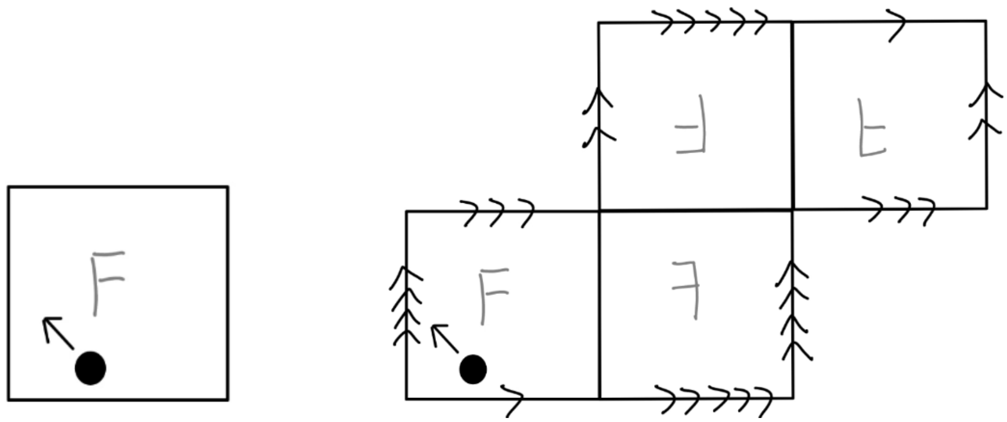
(a)







(b)



(c)

18. Glue all the edges from Problem 9 into a translation surface. Glue them so that any trajectory of a billiards ball on the table agrees with the corresponding trajectory on the translation surface.

19. Glue all the edges from Problem 10 into a translation surface so that any trajectory of a billiards ball on the table agrees with the corresponding trajectory on the translation surface.

20. What do you think should happen if the ball hits the corner of a torus exactly?

21. Complete the unfolding procedure and find the translation surface for a regular pentagon.