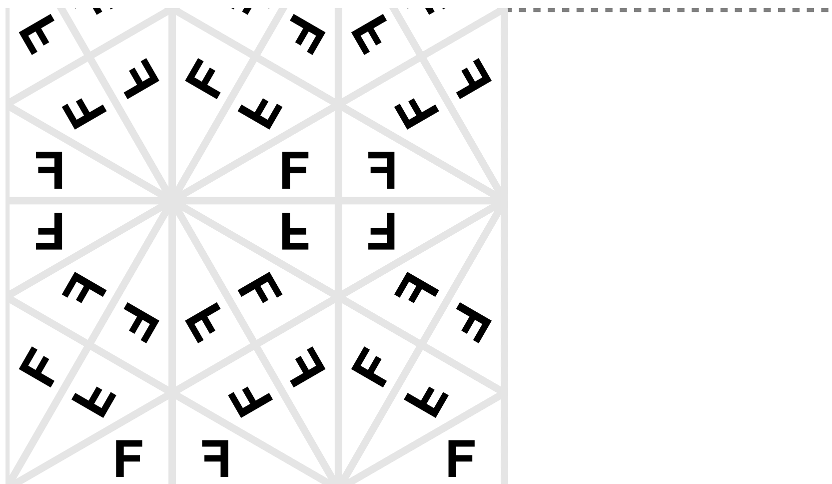
Name: ______________________________

UW Math Circle

Week 24 – Wallpaper Patterns

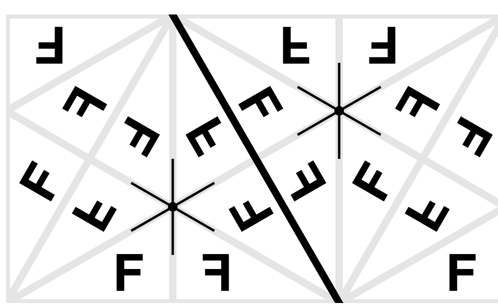
Today, we will talk about wallpapers!

1. This first wallpaper doesn't look very pretty, but it will be useful for us to make some math clear in a bit. Draw in the wallpaper to extend it and fill the box.



2. We're going to analyze the *symmetry* of the wallpaper. The first kind of symmetry we will analyze is *mirror symmetry*. Conveniently, we've already drawn in all the mirror lines in the wallpaper above!

   Whenever mirror lines intersect, we call that a *mirror point*. Two mirror points are *the same type* if you can use other mirror lines to reflect one onto the other. Two examples of mirror points with 3 lines are shown below. They are the same type, by reflecting across the indicated line.

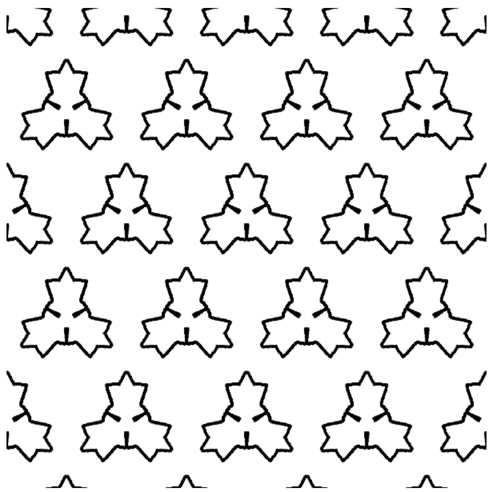


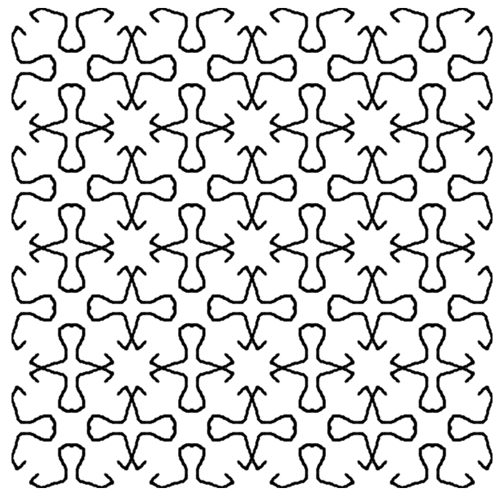
   Mark the mirror points in the wallpaper from question 1. How many types are there? How many lines does each type have?

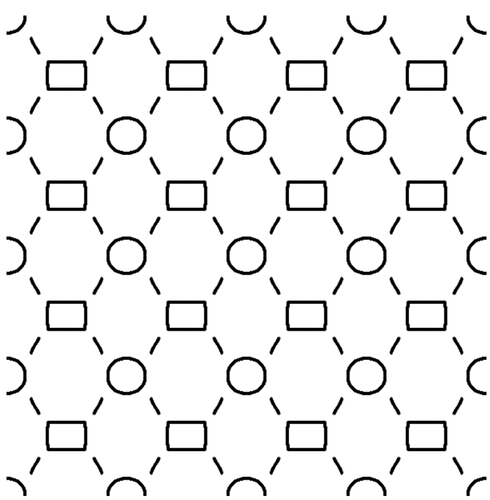
Stop here. Check with an instructor before continuing.

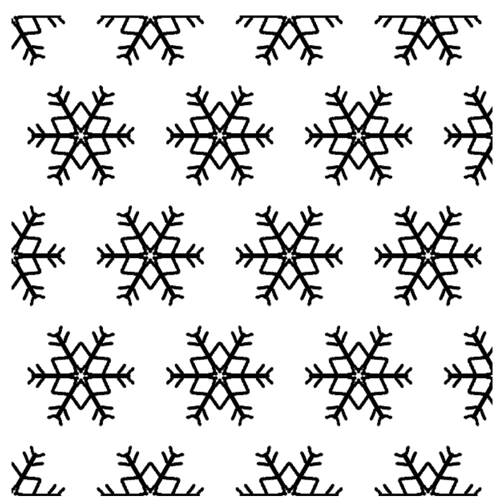
3. For a wallpaper with 3 types of mirror points, one with 6 lines, one with 3 lines, and one with 2 lines (like the one from question 1), we say that the wallpaper has *signature* "$*632$". If there were 3 types, and each type had 3 lines, we would write "$*333$".

   For each of the following wallpapers, mark one mirror point of each type. What is the signature of the pattern? (You'll have to draw the mirror lines yourself!)





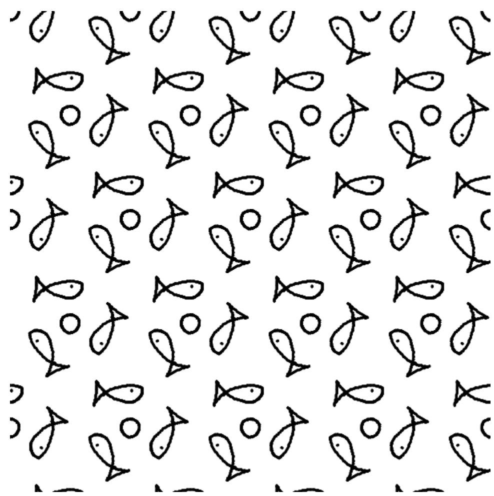




4. Here is a special wallpaper pattern. It also only has mirror symmetry, but we need to give it a special signature: "$**$".

   Draw all the mirror lines, and say why the previous naming scheme didn't work for this.

5. Next, we'll look at *rotational symmetry.* In the picture below, the three fish are rotating around the points marked with a circle $\circ$. We say that the circle is a *rotation point* with 3 turns, because the fish can rotate 3 times before going back to where they started.



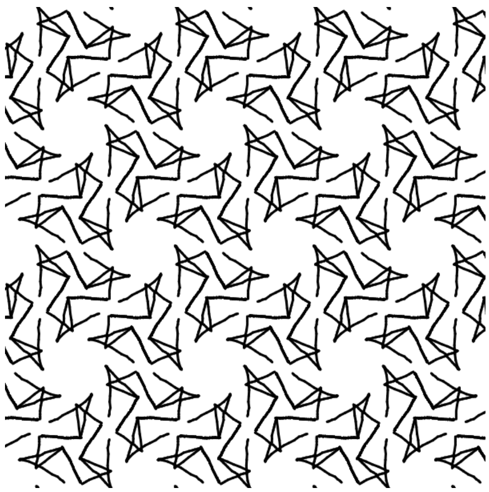
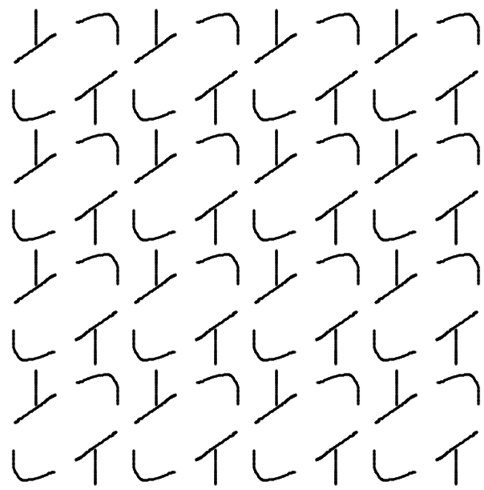
   Spotting if two rotation points are the same type is easy: They are the same if *translating* one point to the other doesn't change how the wallpaper looks at all. "Translate" means move the wallpaper without rotating or reflecting.

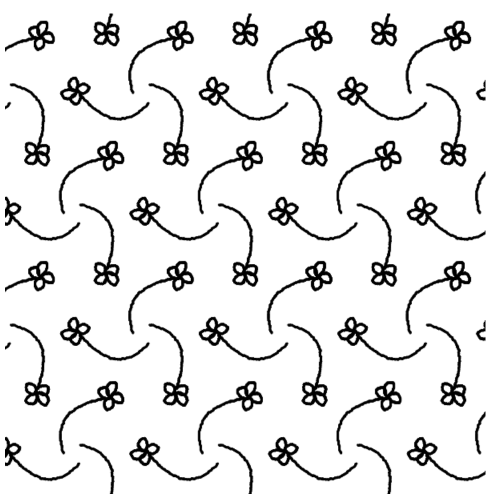
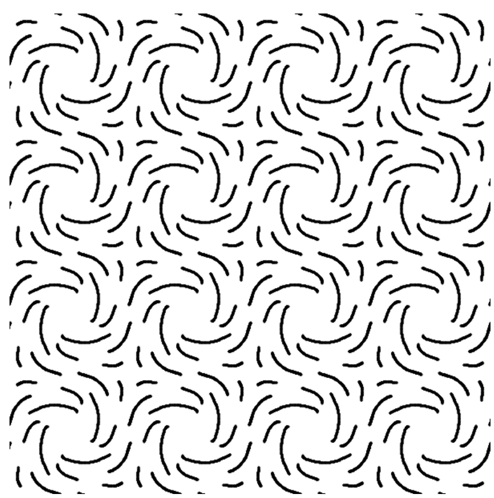
   All of the circles form one type of rotation point. Mark the other rotation points in the picture, and say how many turns each type is. (Hint: there are 2 more types!)

Stop here. Check with an instructor before continuing.

6. For a wallpaper with 3 types of rotation points, each of which has 3 turns (like the one from question 5), we say that the wallpaper has signature “333”. We don’t put a star when it’s just rotation.

   For each of the following wallpapers, mark one rotation point of each type. What is the signature of the pattern?

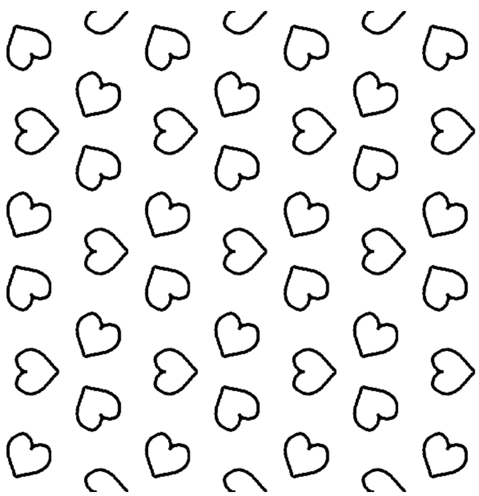



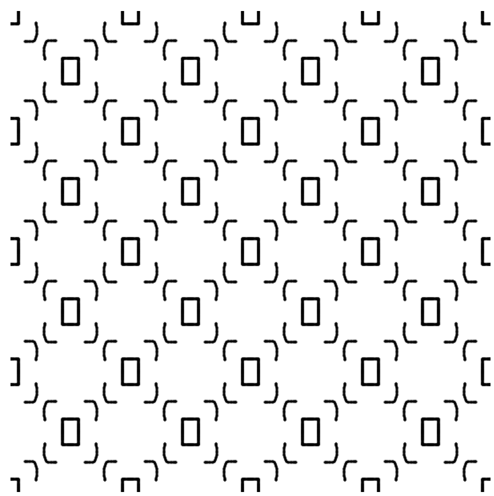
7. Sometimes, a wallpaper can have both mirror and rotational symmetry! When this happens, **find the mirror points first.** When you look for rotation points, don't count any points that are already mirror points.

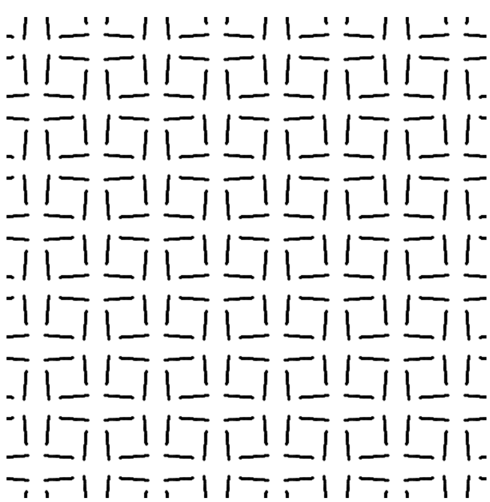
   As for how to write the signature, write the rotation points, then $*$ if there are any mirrors, then the mirror points. For example, if there is one rotation point with 4 turns and one mirror point with 2 lines, you would write "4∗2".

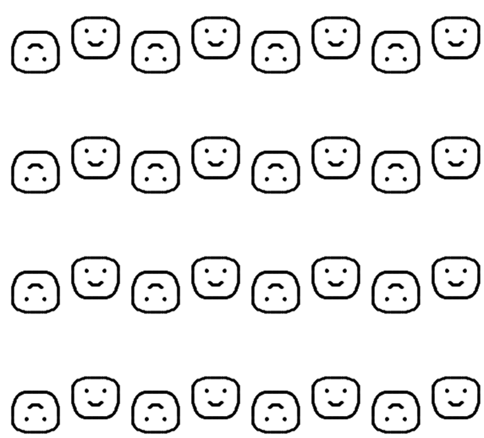
   Note that now with both mirror lines and rotations, two rotation points are now the same type if: either you can translate them, or use mirror lines to reflect one onto the other.

   Do the same as the previous problems for these!

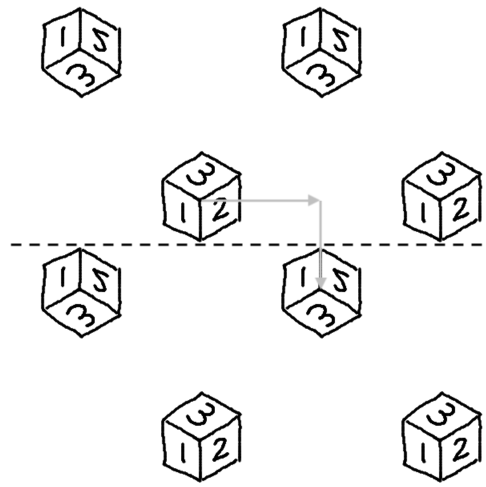








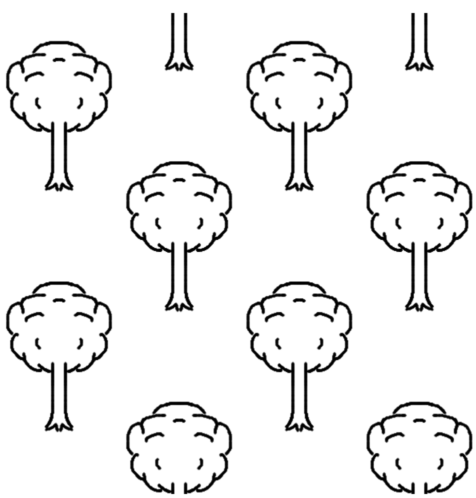
8. The last kind of symmetry is called the *glide*. This one's a little more tricky to see. A glide is a translation followed by a reflection. In the picture below, the cube moves to the right following the gray arrow, and then is reflected across the dashed line. This entire process is called a glide.



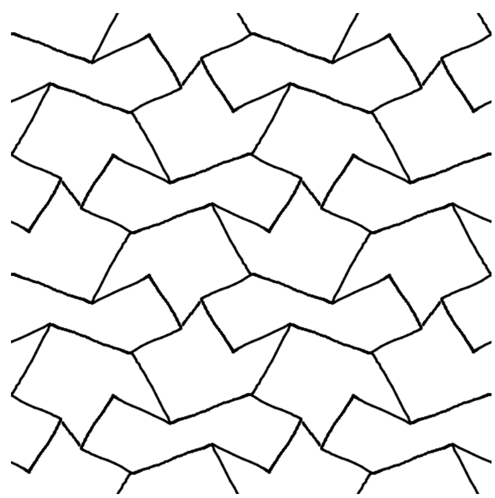
Two glides are of the same type if: you can translate one line of reflection onto the other without changing how the wallpaper looks, or the other symmetries (mirror and/or rotation) can make the lines overlap.

Find and mark another glide in this picture! (The signature of this is "××", because there are two glides.)

9. These last few wallpapers are a little tricky, so I will tell you their signatures. Draw the rotation points, mirror points, and glides on the picture.



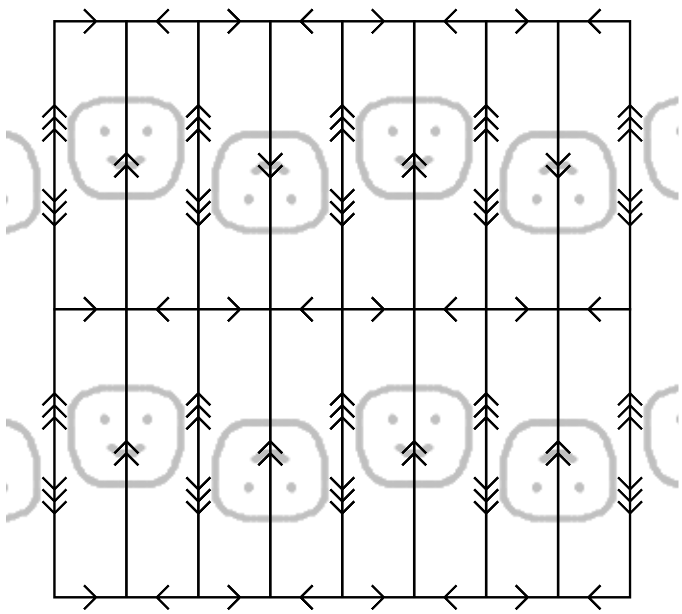
∗× (one mirror line, one glide)



22× (two 2-turn rotations, one glide)

Stop here. Check with an instructor before continuing.

Now, let's investigate how to draw a wallpaper pattern! Each pattern is made up of identical building blocks called *fundamental cells*. For example, for the smiley pattern from Problem 7, a fundamental cell could be any of the skinny rectangles shown below. Note that each of the skinny rectangles has the same design, just possibly flipped, rotated, and translated.



Adding arrows to a fundamental cell makes it a *gluing diagram*. The arrows tell you how to arrange copies of the piece to make the full design. All edges must be lined up (full edge to full edge) so that they **match in number and direction of arrows**.

10. Suppose you were given a stack of the identical pieces shown in the above diagram. Explain how you would turn/flip/glue them together in order to make the original pattern. Use your explanation to come up with rules for how the arrows are determined.

11. Flip back to previous pages and draw the fundamental cells for the following problems. Make sure to use your mirrors, rotations, and glides to make the cell as small as possible! (Note: Patterns may have many possible fundamental cells, even of different shapes!)
    - The treble clefs in Problem 4.
    - The snowflakes in Problem 3.
    - The trees in Problem 9.
    - The hearts in Problem 7.

12. For each of the following, figure out the fundamental cells by looking at a previous example from earlier in today's packet. Then use the fundamental cells to help you draw your own wallpaper of that signature.

$*632$

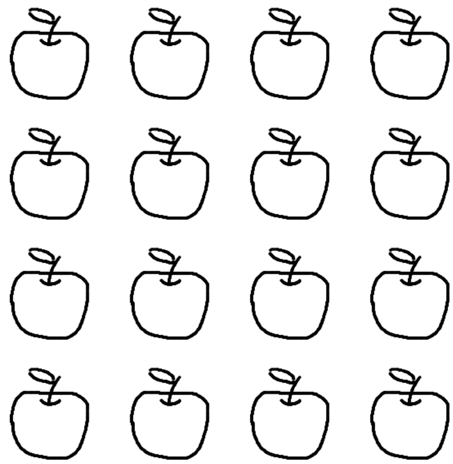
$3{*}3$

$4{*}2$

$22\times$

13. Pick any signature and draw your own wallpapers! Then swap with someone nearby and challenge them to identify what the signature is.

By the way, you've seen 16 different kinds of wallpapers now. There's also the wallpaper that involves doing no fancy symmetries, just translating the same picture over and over. It looks like this:



It turns out that these 17 types wallpaper patterns are the only ones to exist! This result is known as **Conway's Magic Theorem**. (Conway wasn't the first person to prove this theorem — that was Evgraf Fedorov in 1891 — but he popularized it and gave it a cool name, so everyone calls it Conway's theorem now.)

To summarize, here are all 17 possible signatures of wallpapers:

- $\circ$ (no symmetries)
- $*632$
- $*442$
- $*333$
- $*2222$
- $**$
- 632
- 442
- 333
- 2222
- $4{*}2$
- $3{*}3$
- $2{*}22$
- $22*$
- $*\times$
- $22\times$
- $\times\times$