Name: ______________________________

UW Math Circle

Week 22 – Coloring the Plane

1 Map Coloring

You have a map divided into regions, and you need to color each region a single, solid color. You can reuse colors, but two adjacent regions can't be the same color. How many different colors do you need?

(Note: if two regions only meet at a corner, they don't count as adjacent—so it's fine if Utah and New Mexico are the same color.)

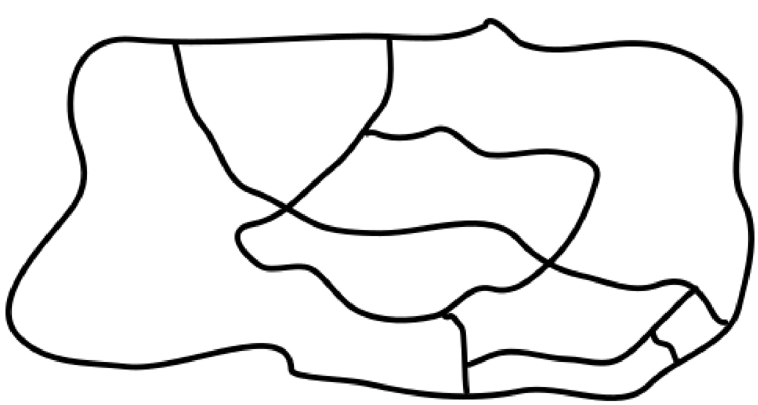
The *four color theorem* says that **four colors are enough to color any map!**

Of course, that doesn't mean you always need four colors. For each problem 1-5,

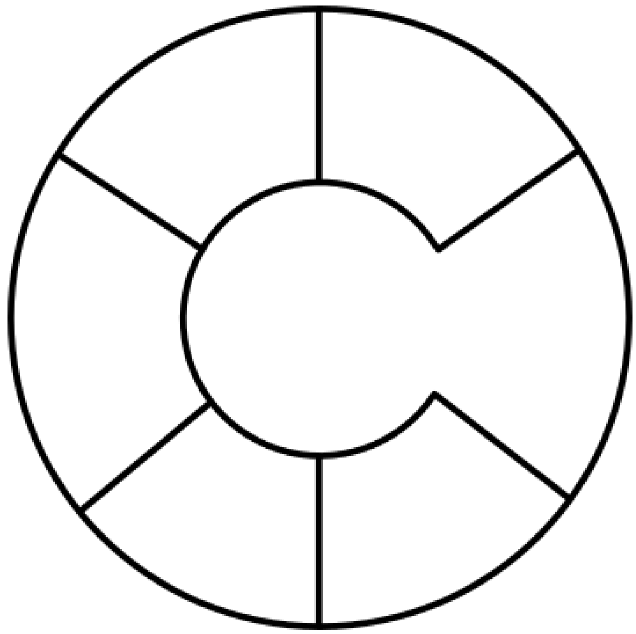
i. **Color the map in as few colors as you can!**

ii. **Are you sure you're using the fewest colors possible? How do you know?**

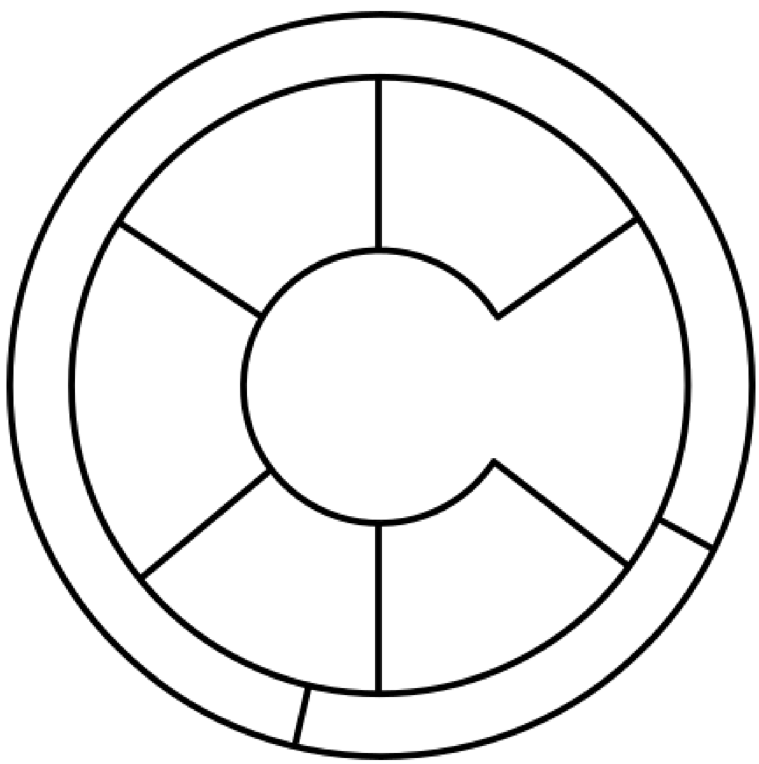
1.



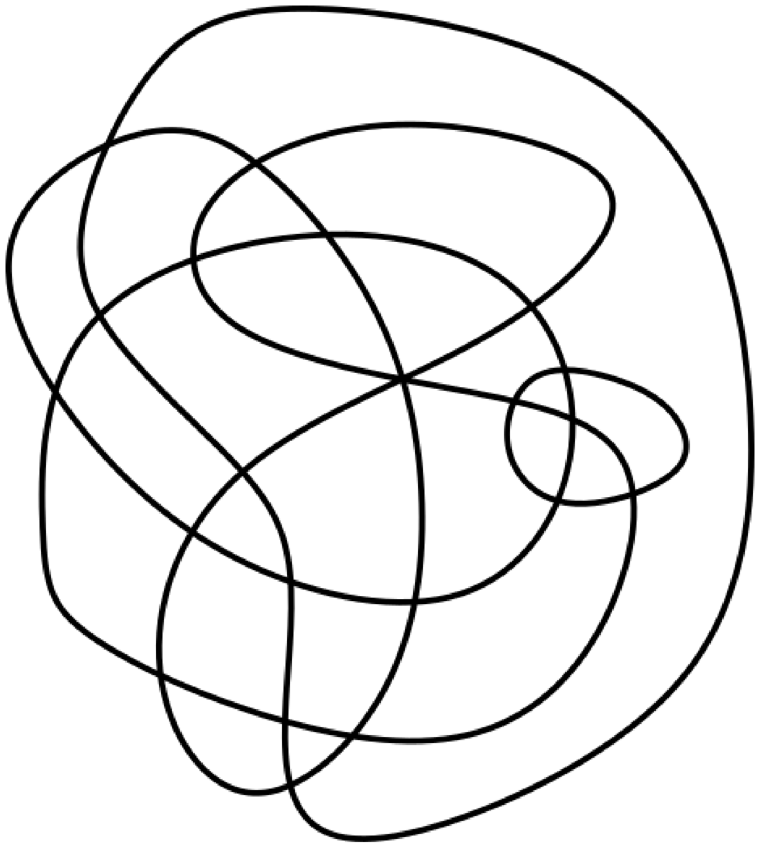
2.



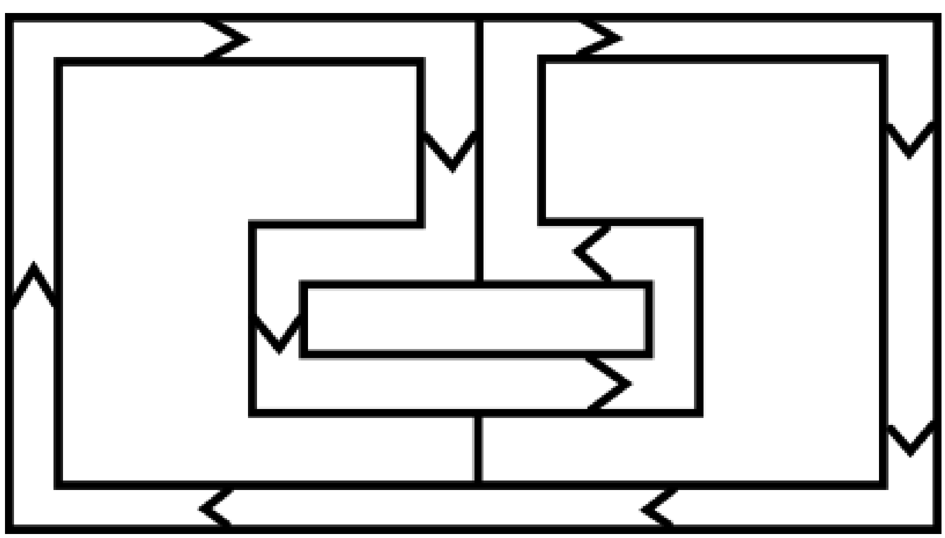
3.



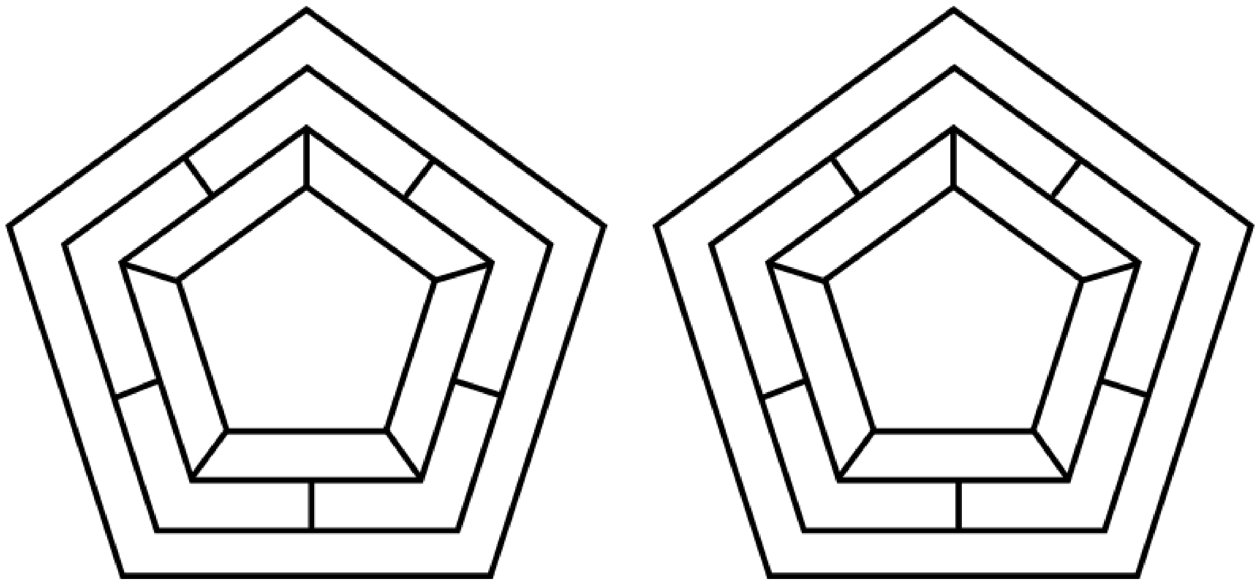
4.



5.



6. Below are two identical maps. Color the first map with **twelve different** colors. Can you color the second map with the same twelve colors, but arranged differently, so that any two colors which are adjacent in the first map are *not* adjacent in the second map?



(If you don't have twelve different colors, you can label the regions with numbers instead.)

7. *Does the four color theorem work in three dimensions?* Suppose you have a three-dimensional space divided into solid regions. You want to color them so that no two adjacent regions (i.e. regions sharing part of their surface area) are the same color. Are four colors always enough?

   If so, why? If not, is there some larger number of colors that's guaranteed to be enough?

8. You have a map of Ozma, an alien planet with **ten continents**. Each continent is divided into regions, and you need to color the map so that no two adjacent regions are the same color.

   But watch out—planet Ozma has dramatic **continental drift**! In the future, two continents may collide unexpectedly, and some regions on the coast of each continent will become adjacent. You must color your map so no matter how the continents collide, any two adjacent regions will still be different colors.

   How many different colors do you need to guarantee you can color the map?

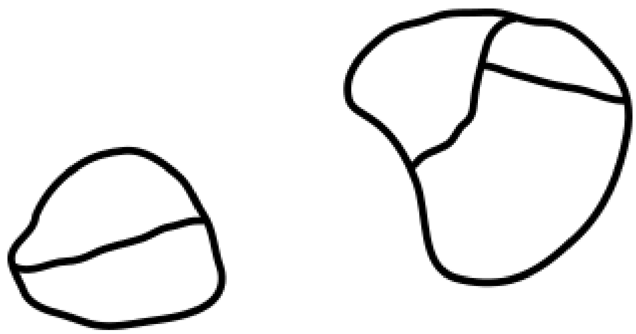


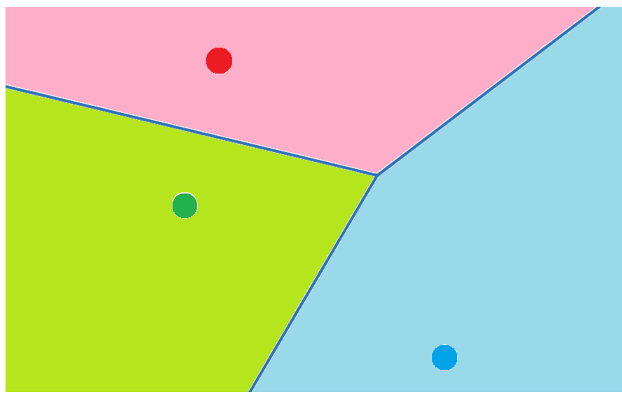
Figure 1: example map

   (For example, you need *five colors* for the map above, since any region of the first continent may collide with any region of the second.)

2 Voronoi Diagrams

Let's try a different way of coloring the plane! To draw a map with n regions, we'll start by choosing n different points to be the regions' "capitols". Then, we'll divide up the plane based on which capitol is the closest.

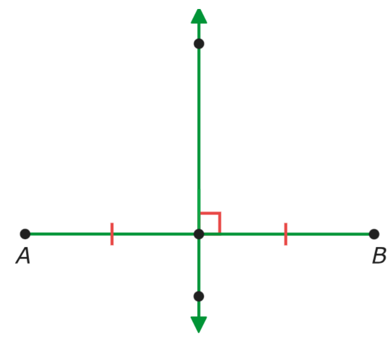
Below, we've chosen three points as our capitols (Red, Green, and Blue). To color the rest of the plane, we use red at any location that's closest to the Red capitol, green closest to the Green capitol, and blue closest to the Blue capitol.



This way of coloring the plane is called a *Voronoi diagram*!

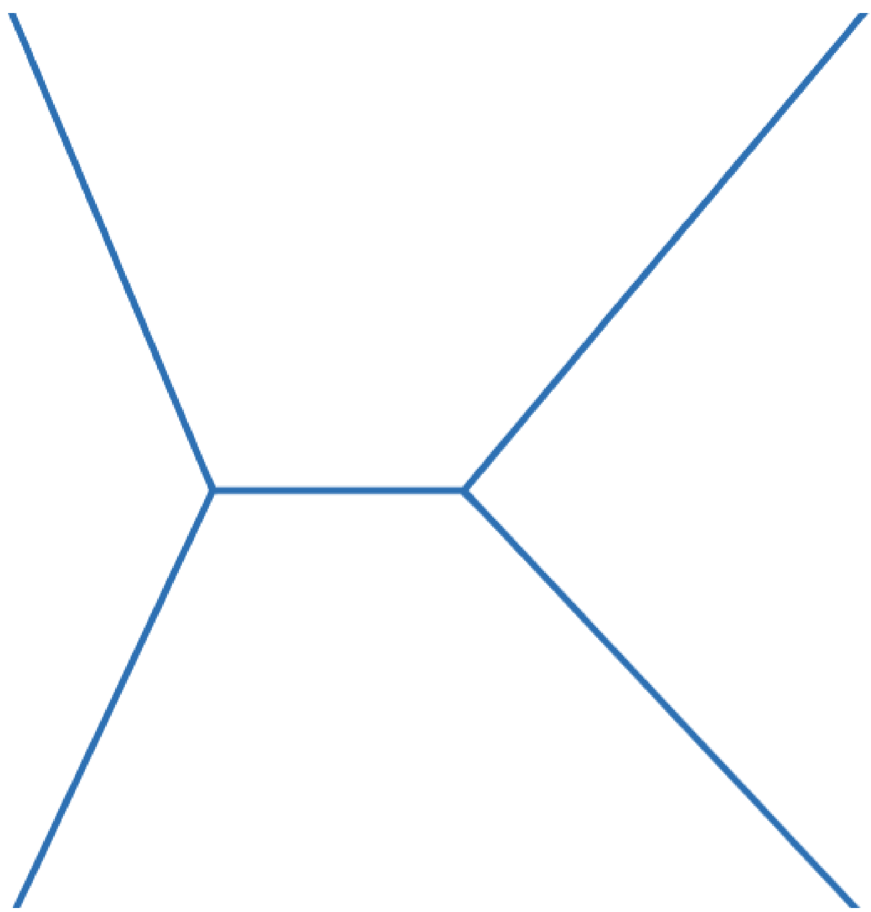
9. Sketch the Voronoi diagram with these four capitols.

10. If A and B are points in the plane, any third point that's equidistant from A and B will lie on a line perpendicular to segment AB, cutting halfway between A and B. This line is called the *perpendicular bisector* of AB.

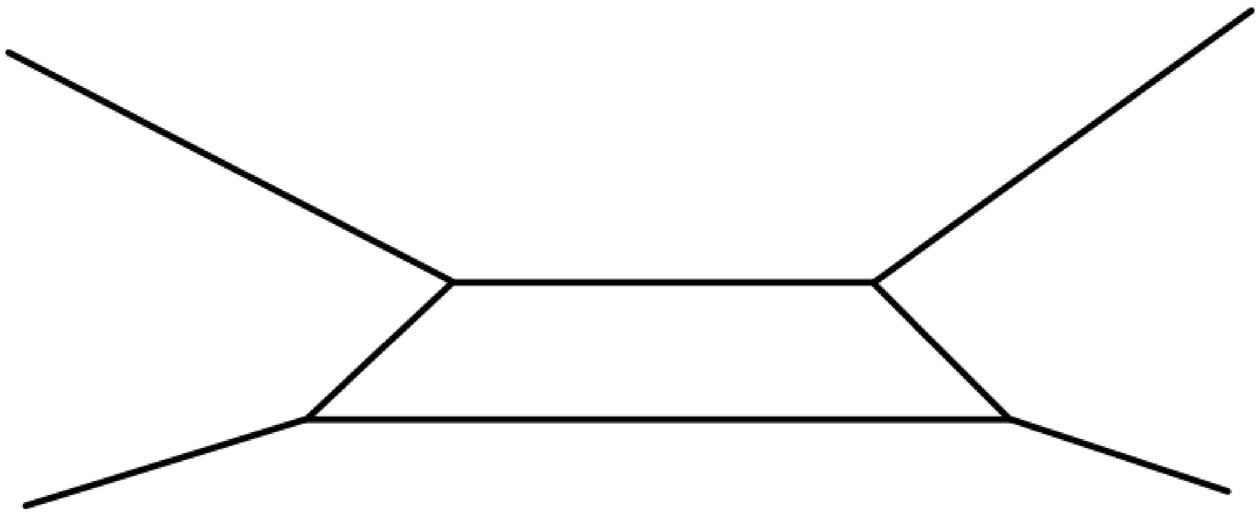


Using this, explain why the border between two regions in a Voronoi diagram is always perpendicular to the line between their capitols.

11. This is another Voronoi diagram, but the capitols aren't shown. Using the diagram, try to determine where the capitols are! (As closely as you can—it won't be exact.)



12. Is it possible to choose five capitols that result in this Voronoi diagram? Why or why not?



13. Is it possible to choose five capitols that result in this Voronoi diagram? Why or why not?