Name: ______________________________

UW Math Circle

Week 21 - Conjectures & Counterexamples

(Adapted from the mathforlove.com Counterexamples Lesson)

1 Counterexamples

A **counterexample** is an example which demonstrates why a statement is incorrect. For example, the statement "All prime numbers are odd." is false since 2 is a counterexample.

Let us find some of our own counterexamples! Prove the following statements about shapes wrong by providing a counterexample.

1. A polygon with 4 sides is a square.

   Counterexample:

2. If the perimeter of a rectangle is not a whole number then its area is also not a whole number.

   Counterexample:

3. There is no triangle with the same area as a square.

   Counterexample:

4. Increasing all three side lengths of a triangle will always make the area larger.

   Counterexample:

Prove the following statements wrong by providing a counterexample.

5. Alice and Bob each own a lot of cats. If we know that

   - Alice is older than Bob
   - The average age of Alice's cats is greater than the average age of Bob's cats

   then the average age of *Alice and all her cats* must be greater than the average age of *Bob and all his cats.*

   Counterexample:

6. Every English word with six letters or more uses at least one letter from the first half of the alphabet.

   Counterexample:

7. You're in the middle of a hedge maze with one exit. Unless the maze is impossible, you are guaranteed to reach the exit eventually if you just follow the right-hand wall.

   Counterexample:

8. You give a gumball machine D dollars. In return, the machine gives you $a(D)(D-1)$ gumballs for some nonzero number a. If every time you give the machine a whole number of dollars you get a whole number of gumballs, then a must be a whole number.

   Counterexample:

Fill in the counterexample blanks:

9. Farts are only caused by humans and animals.

   Counterexample: W__□□__ee __u □__□__ __

10. Trying harder always makes puzzles easier.

    Counterexample: □ing __□ Tr__ __

11. Fruit does not cause people to slip.

    Counterexample: __□__ __ __ __ □__ __□$_{\times 2}$

Unscramble the letters in boxes from the last three questions to give a counterexample:

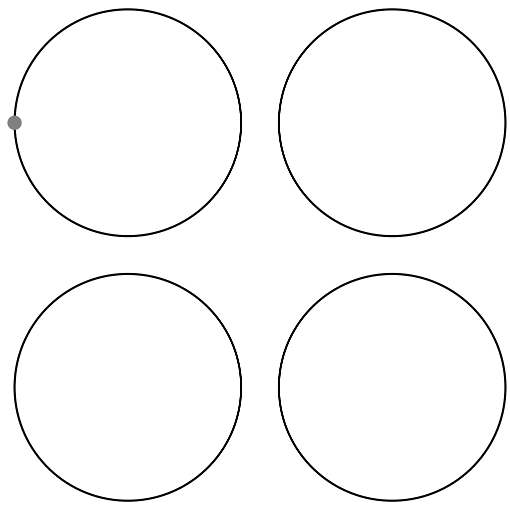
12. Holidays do not contain pranks.

    Counterexample: □□□□□ □□□□□

2 Conjectures Based on Patterns

Patterns in math often help us to guess what to expect. A guess that has not yet been proven is called a **conjecture**. In the next few questions, we will observe some patterns, make conjectures, and then see if we can prove the conjecture or find a counterexample to disprove it!

1. **Moser's Circle Problem:** Draw n dots on a circle. Connect each dot to all the others by a straight line then count how many regions the circle is broken up into.



n	# Regions
1	1
2	
3	
4	

Make a **conjecture** about how many regions there will be with 5 dots? n dots?

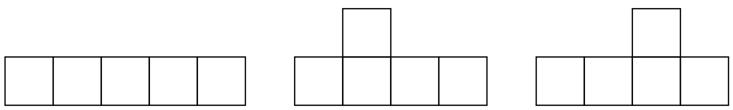
2. Prove your conjecture from Problem 1 or give a counterexample to the conjecture.

3. When stacking boxes one after another, they will not topple as long as:

   - The first and last stack has only one box.
   - The height of neighboring stacks only differs by 1.

   How many ways can you draw n stacked boxes so that they will not topple?

   For example, with 5 boxes you can stack them 3 different ways:



n	**# nontoppling stackings**
2	
3	
4	
5	3
6	
7	

   Make a **conjecture** about how many nontoppling stackings will there be with 8 boxes? n boxes?

4. Prove your conjecture from Problem 3 or give a counterexample to the conjecture.

5. How many 3 digit numbers have digits which sum to n?

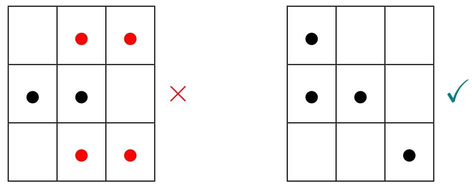
   For example, 100 is the only 3 digit number whose digit sum to 1.

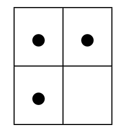
n	# 3 digit numbers whose digits sum to n
1	1
2	
3	
4	
5	

   Make a **conjecture** about how many 3-digit numbers there with digits summing to 6? to n?

6. Prove your conjecture from Problem 5 or explain why and for what n the pattern will break.

7. On an $n \times n$ grid, place dots so that no four dots form a rectangle when connected by lines. What is the maximal number of dots we can have?



For example, on a 2×2 grid, the maximal number of dots is 3: 

n	**Maximal # of dots**
2	3
3	
4	
5	

Make a **conjecture** about what the maximal number of dots is for a 6×6 grid? $n \times n$?

8. Prove your conjecture from Problem 7 or give a counterexample to the conjecture.

9. Compute the sum of the first n odd numbers.

   For example, the sum of the first 5 odd numbers is $1 + 3 + 5 + 7 + 9 = 25$.

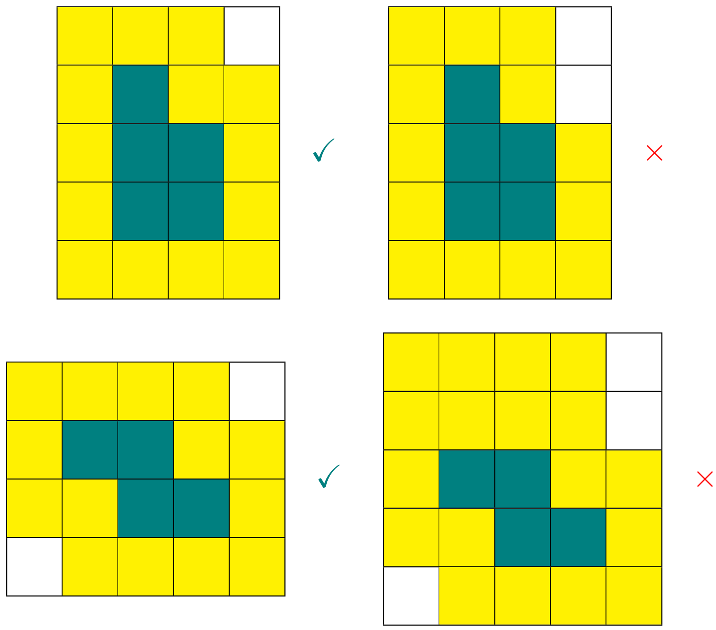
n	**Sum of first n odd numbers**
1	
2	
3	
4	
5	25
6	

   Make a **conjecture** about the sum of the first n odd numbers!

10. Prove your conjecture from Problem 9 or give a counterexample to the conjecture.

3 Make Your Own Conjectures!

On grid paper, an island is modeled by a collection of connected forest squares surrounded by beach squares. Beyond the beach is the ocean. The beach is only one square thick and the forest is not allowed to touch the ocean. Here are some examples and nonexamples:



1. On your grid paper, try drawing some islands by coloring forest squares one color and beach squares another. Then, write down some conjectures about islands!

2. Share your conjectures in your group. Then, try and prove/find counterexamples to the conjectures made!
   (If a counterexample is found, try to alter your conjecture and repeat the process!)