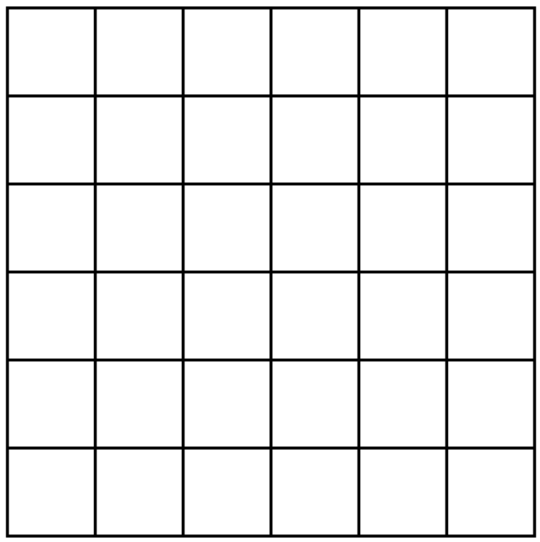
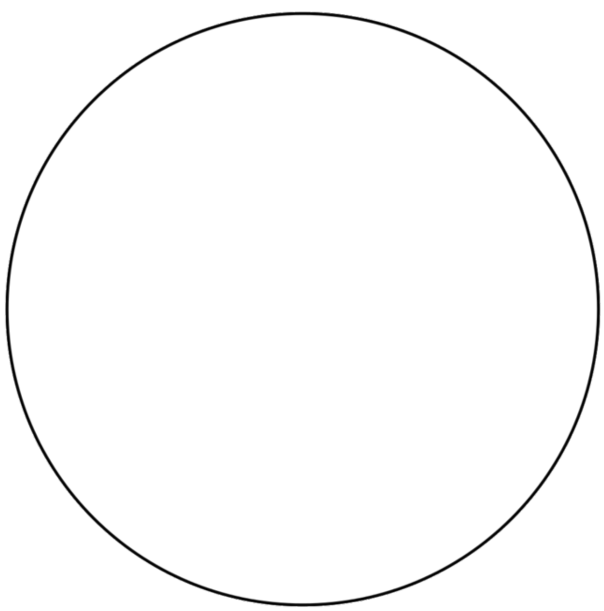
UW Math Circle

Week 2

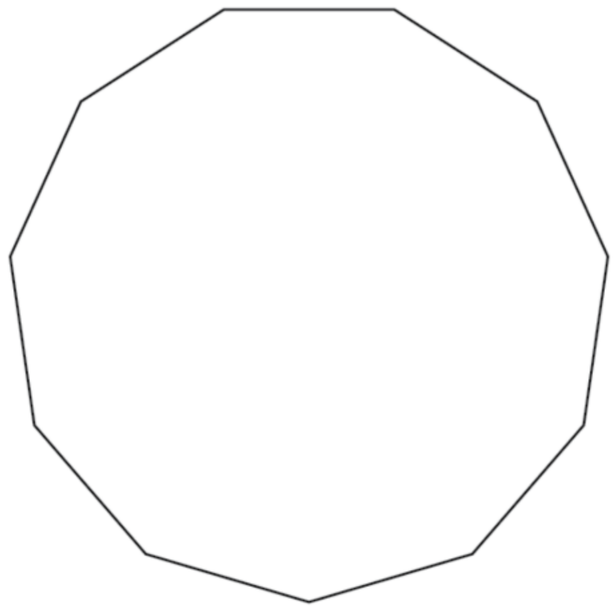
1. This game is played on a 6×6 grid. Two players take turns placing a 1×2 domino on the grid, so it covers two adjacent squares which have not been covered already. The player to make the last legal move wins.



2. This game is played on a circular board. Two players take turns placing a penny flat on the board. Pennies must lie entirely within the circle and may not overlap each other. The player to make the last legal move wins.



3. This game is played on a regular 11-gon. Two players take turns drawing a diagonal of the 11-gon (a straight line connecting two non-adjacent corners). Several diagonals may share an endpoint, but the same diagonal may not be drawn twice, and diagonals may not cross. The player to make the last legal move wins.



4. Two players start with a pile of 100 stones and take turns removing stones. Each turn, the number of stones removed must be a power of two (1, 2, 4, 8, 16, 32, or 64). The player to remove the last stone wins.

5. Two players start with a pile of 100 stones and take turns removing stones. Each turn, the number of stones removed must be a proper divisor of the total number in the pile. The player to make the last legal move wins (leaving one stone in the pile). Below is an example game.

Start: **100**.
Player 1: $100 - 20 = \mathbf{80}$.
Player 2: $80 - 16 = \mathbf{64}$.
Player 1: $64 - 32 = \mathbf{32}$.
Player 2: $32 - 16 = \mathbf{16}$.
Player 1: $16 - 8 = \mathbf{8}$.
Player 2: $8 - 4 = \mathbf{4}$.
Player 1: $4 - 2 = \mathbf{2}$.
Player 2: $2 - 1 = \mathbf{1}$. Player 2 wins!

Let's shift gears a little: did you notice that a lot of the games you were playing these past 2 weeks had a very similar feeling to them? In the 1930s, two mathematicians named Roland Sprague and Patrick Grundy figured out this feeling was no coincidence!

6. A game is called *impartial* if the allowed moves depends only on the current game board state, and not which player is moving. A game is called *finite* if the number of possible moves never infinity. A game is called *normal play* if the last legal move wins, and *misère play* if the last moves loses.

   Which of the games that you played (both this week and last week) are impartial? Finite? Which are normal play, and which were misère play?

7. The following impartial, finite, normal play game is called Nim:

   There are some stacks of coins (at least one stack), and players take turns removing coins from the stacks. In each turn, players must remove at least one coin, and may remove as many coins as they like from a single stack. The last person to make a legal move wins.

   Try out a few games of Nim with a partner with different starting stacks!

   (a) Three stacks of 3, 4, and 5 coins.
   (b) Two stacks of 7 and 9 coins.
   (c) For what stack sizes does the first player win two-stack Nim?
   (d) Who should be the winner when you start with just one stack of zero coins?

(continued on the back)

What does it really mean for two games to be the same? There's no right answer! But if we want to determine whether two games *are* the same, we need to fix a definition. Focusing on Nim first, Sprague and Grundy suggested this: Two games are equivalent if, whenever you add a new pile of coins of the same size to both games, both games are a winning position for the same player.

Also, to make my life easier, I will write the game in 7a as (3, 4, 5), 7b as (7, 9), 7d as (0), and similarly for other Nim games.

8. Explain why (12, 12) Nim is equivalent to (0) Nim.

9. Explain why (5) Nim is not equivalent to (8) Nim.

The amazing theorem of Sprague and Grundy is that every impartial, finite, normal play game is equivalent to one-stack Nim game! Let's see this in action.

10. Recall #5 from last week's worksheet:

    A rook starts in the top-right corner of an 8x8 chessboard. Two players take turns moving the rook either left or down as many squares as they'd like. The player to move the rook to the bottom-left square wins.

    Which one-stack Nim game is this equivalent to?

11. Recall #1 from last week's worksheet. I've modified it a tiny bit:

    The game starts with a pile of 10 coins. Two players take turns removing either 1 or 2 coins from the pile. The player who takes the last coin wins.

    Which one-stack Nim game is this equivalent to? (Hint: First explain why this game starting with 3 coins is equivalent to this game starting with 0 coins.)