Week 8

Question 1. Simplify these numbers:

(a) $3 + \frac{1}{7}$

(b) $2 + \cfrac{1}{3 + \cfrac{1}{5}}$

(c) $4 + \cfrac{1}{8 + \cfrac{1}{3 + \cfrac{1}{2}}}$

(d) $3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}$

Question 2. A number written in this pattern:

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cdots}}}$$

is called a "continued fraction". Every number can be written as a continued fraction! For example, let's try $\frac{47}{21}$. It's a little bit more than 2, so we can write

$$\frac{47}{21} = 2 + \text{a bit} = 2 + \frac{5}{21} = 2 + \frac{1}{21/5}.$$

Now, $\frac{21}{5}$ is a little more than 4, so we can write

$$\frac{47}{21} = 2 + \frac{1}{21/5} = 2 + \frac{1}{4 + \text{a bit}} = 2 + \cfrac{1}{4 + \cfrac{1}{5}}$$

And since the numerator of $\frac{1}{5}$ is 1, we stop here.

Try this yourself with these fractions:

$$\frac{10}{7}, \qquad \frac{65}{29}, \qquad \frac{559}{504}, \qquad \frac{413}{134}, \qquad \frac{134}{413}, \qquad \frac{205}{46}$$

So far, we've only found continuous fraction versions of numbers that were already fractions to start with. But we can do this for other numbers too! For example, π is a little bit more than 3:

$$\pi = 3 + 0.14159\ldots = 3 + \frac{1}{7.0625\ldots}$$

and $7.0625\ldots$ is a bit more than 7:

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15.997\ldots}}$$

and so on:

$$= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}} = [3, 7, 15, 1, 292, \ldots].$$

Question 3. Compute the first few terms of the continued fraction versions of:

(a) $e = 2.7182818284590\cdots$

(b) $\sqrt{2} = 1.414213562373\cdots$

(c) $\sqrt{6} = 2.449489742783\cdots$

(d) $\sqrt[3]{3} = 1.442249570307\cdots$

Question 4. What do you get if you only take the first few steps in a number's continued fraction? For π, for example, what happens if you just take $3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}$, or $3 + \cfrac{1}{7 + \cfrac{1}{15}}$, or $3 + \frac{1}{7}$? It might help to see the pattern if you write these numbers as decimals. Try this for some of the other continued fractions you've calculated too.

Question 5. The Fibonacci numbers are the sequence of numbers starting with $1, 1$ where each number is the sum of the two previous ones:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Try dividing each number in the sequence by the one before it. Do you notice any patterns?

Question 6. The Lucas numbers are another Fibonacci-type sequence. They're defined the same way, except that they start with $2, 1$ instead of $1, 1$:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$$

Calculate the ratios of consecutive Lucas numbers, like you did for the Fibonacci numbers. Any patterns here?

Question 7. Now make your own Fibonacci-type sequence, by picking two starting numbers and filling in the rest of the sequence by adding the two previous numbers together at each step. Are there patterns in the ratios? Does it depend on what starting numbers you choose?

Question 8. For the Fibonacci numbers, calculate the continued fraction versions of the ratios. What do you notice?

Question 9. The seeds at the centre of a sunflower form spirals, and if you count them, you'll often find that the number of spirals is a Fibonacci number. Something similar happens with the spikes on a pineapple, and the bumps on a pinecone. Let's investigate why this might happen!

Visit this link: `https://www.desmos.com/calculator/fk5pqg1xgg`

This picture illustrates how a plant might arrange its seeds. The idea is that there is something at the centre of a flower that creates a seed, then turns x degrees to face a new direction, creates another seed in that new direction, turns x degrees again, and so on. The variable q is the turning angle: $q = 1$ represents a full turn, $q = \frac{1}{2}$ represents a half turn, and so on.

Try entering some different values for q. What patterns do you see? Which values of q make the seeds spread out more evenly? What relationships can you see between the pattern and q's continued fraction representation?