Week 7

Question 1. This is a magic square:

4	9	2
3	5	7
8	1	6

A magic square is an arrangement of numbers in a square grid, with these rules:

I. The sum of the numbers in each row and column is the same,

II. The diagonals also sum to the same thing,

III. All the numbers in the grid are different, and

IV. Every number from 1 to the number of squares in the grid appears once.

(a) Find all the 1×1 magic squares. *(Hint: there aren't very many.)*

(b) Now find all the 2×2 magic squares. *(Hint: there aren't many of these either.)*

(c) Can you find any other 3×3 magic squares? *(Hint: What does the row sum have to be? What about the number in the middle square?)*

(d) *(Challenge.)* Can you find a 4×4 magic square (without googling it)?

Here are the rules again:

I. The sum of the numbers in each row and column is the same,
II. The diagonals also sum to the same thing,
III. All the numbers in the grid are different, and
IV. Every number from 1 to the number of squares in the grid appears once.

Question 2. It's a bit difficult to find 2×2 magic squares, so let's make it easier by getting rid of some of the rules.

(a) How many 2×2 magic squares are there if we only use rules I., III. and IV.?

(b) What if we only use rules I. and III.?

(c) Or rules I. and II.?

(d) Or just rule I.?

Question 3. Now try to look for 4×4 magic squares, using your choice of rules. Which combinations of rules make this easier or harder? (Make sure you always use rule I..)

Question 4. Instead of regular arithmetic, let's try to find magic squares using Arithmetic Modulo 2. Here's how that works, if you're not familiar with it:

- The only numbers you need to use are 0 and 1.
- Here are the addition rules:

$$0+0=0, \qquad 0+1=1, \qquad 1+0=1, \qquad 1+1=0.$$

Got it? Great! Let's practise. Calculate these:

$$1+1+1=\_\_\_\_ \qquad 0+0+0+0=\_\_\_\_$$

$$0+0+1+1+0=\_\_\_\_ \qquad 1+0+0+0+1=\_\_\_\_$$

$$0+1+1+0+1+1=\_\_\_\_ \qquad 1+1+1+1+0+1+1+1=\_\_\_\_$$

$$1+0+1+0+1+1+0+1+1+1+0+1=\_\_\_\_$$

$$1+1+0+1+0+1+1+1+0+1+0+1+0+0+0+1=\_\_\_\_$$

(Is there an easy way to do this? *Hint:* how many 1s are there in each sum?)

Question 5. Now, let's get back to magic squares! Let's just use rule I. (the rule about row and column sums), to make it simple. Here's an example of a magic square modulo 2:

1	1	0	0
0	1	0	1
1	1	0	0
0	1	0	1

(Did we make a mistake? Start by checking that this really is a magic square...)

How many 2×2 magic squares are there? How many of them have rows that sum to 0, and how many of them have rows that sum to 1?

How many 3×3 magic squares are there where the rows sum to 0, and how many have a row sum of 1?

What about 4×4, or 5×5, or 6×6, or...

If you can figure out these numbers, do you notice any patterns? What fraction of all grids filled with 0s and 1s are magic squares?

Question 6. Let's play a game!

Mewtwo and Alakazam have some cards with the numbers 1 to 9 written on them, and on each player's turn, they take one of the cards. If a player has collected three cards that sum to 15, they win. If there are no cards left and no one has a set of three that adds to 15, the game is a draw.

For example: Mewtwo goes first, and takes the card numbered **8**. Next, Alakazam takes card **5**, then Mewtwo takes number **2**. Then Alakazam takes **6**, Mewtwo takes **4**, and Alakazam takes **3**. But then Mewtwo takes card number **9**, and wins: Mewtwo's cards are **8**, **2**, **4** and **9**, and $\mathbf{2} + \mathbf{4} + \mathbf{9} = \mathbf{15}$.

Try playing this game with a friend!

Why did we put this game on a worksheet about magic squares?