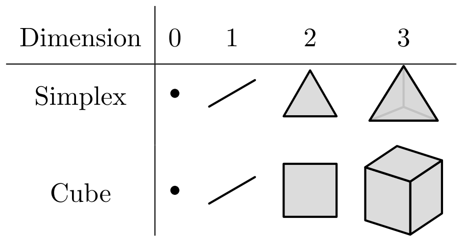
Week 8

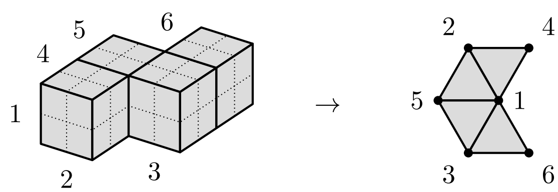
Here's what we've learnt over the last few weeks:

- A "simplex" is a point, a line segment, a triangle, a tetrahedron, or a higher-dimensional shape. A "cube" is a point, a line segment, a square, a 3D cube, or a higher-dimensional shape.



- We can build a "simplicial complex" by attaching simplices along their sides, or a "cubical complex" by attaching cubes along their sides.

- Every *cubical* complex has a corresponding *simplicial* complex, called its "crossing complex": it has a vertex for each *hyperplane* (i.e. a slice through the cubical complex that cuts some of the cubes exactly in half), and it has a simplex for each set of vertices whose corresponding hyperplanes all cross each other.



- If you calculate this formula for the crossing complex and expand everything out:

$$1 + (\#\text{vertices}) \cdot (s+1) + (\#\text{edges}) \cdot (s+1)^2 + (\#\text{triangles}) \cdot (s+1)^3 + (\#\text{tetrahedrons}) \cdot (s+1)^4$$

  then it looks like the coefficients are the numbers of faces of the original cubical complex.

  (Note: This doesn't work for all cubical complexes — all the ones we've looked at in the last two weeks are a special type called "CAT(0) cubical complexes", and the formula does work for them.)

Today, we'll try to prove this observation!

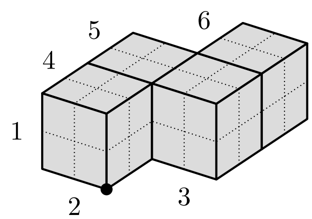
Question 1. First, let's examine the formula $(s+1)^n$.

Expand this out for $n = 0, 1, 2, 3\ldots$, using Wolfram Alpha to help if you want. Do you recognise the coefficients?

(Hint: We studied them in Math Circle several months ago!)

(Challenge.) Explain why these numbers are here.

Question 2. Here's a cubical complex. I've drawn a dot on one of the vertices — think of this vertex as "home".

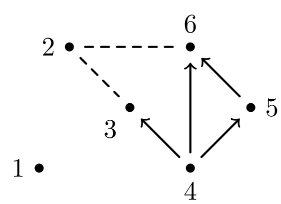


Imagine you live at the "home" vertex, and you want to go on a journey through the cubical complex. This journey might cross some of the hyperplanes, and not cross others.

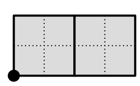
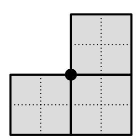
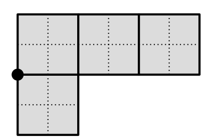
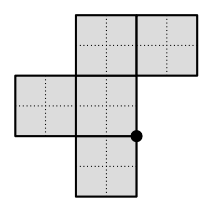
Let's draw a diagram representing the relationships between the hyperplanes. Here are the rules:

- If you have to cross hyperplane A before you can reach hyperplane B, draw an arrow from A to B.
- If it's impossible to cross both hyperplanes A and B once each on a single journey, draw a dotted line between A and B.

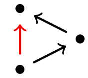
For example, here are the hyperplane relationships for this cubical complex:



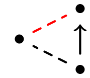
Draw a hyperplane relationship diagram for the following cubical complexes.

Question 3. These hyperplane relationship diagrams have some properties. Can you explain why?

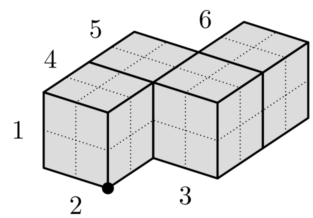
(a) If we have these black arrows, we must also have this red arrow: 

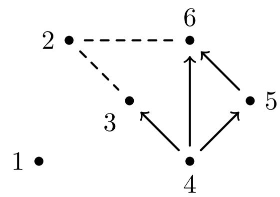
(b) We can't have arrows like this: 

(c) If we have this black arrow and line, we must also have this red line: 

In 2012, Federico Ardila, Megan Owen and Seth Sullivant proved that these properties work both ways: every hyperplane relationship diagram has these properties, but also, any diagram with these properties is a hyperplane relationship diagram for some CAT(0) cubical complex!

Question 4. Here's the hyperplane relationship diagram from the last page again:





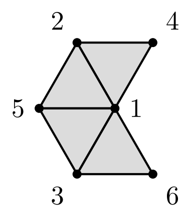
Label each vertex in the cubical complex with the list of hyperplanes you need to cross to travel to it.

Consider each label as a subset of the hyperplane relationship diagram. Check that these subsets have this property:

Property 1: *If you start in the subset and follow some arrows in reverse, you stay in the subset.*

Explain why. Are there any subsets with this property that aren't the labels for any vertices? (Don't forget the vertices in the "back" of the picture.)

Question 5. Here's the crossing complex for this cubical complex:



Consider the vertex sets for the simplices in the crossing complex (e.g. the triangle $\{1, 2, 5\}$, the edge $\{3, 6\}$, the vertex $\{4\}$, ...). Look at these sets in the hyperplane relationship diagram, and check that they all have this property:

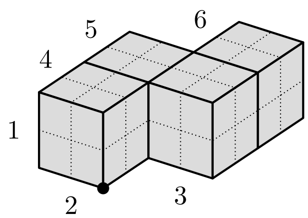
Property 2: *No two elements of the subset are connected by an arrow or a dotted line.*

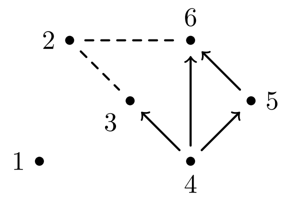
Explain why. Are there any subsets with this property that aren't the vertex set of a simplex in the crossing complex?

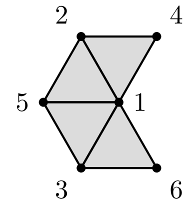
Question 6. Can you see a relationship between the sets in questions 4 and 5?

(Hint: If you take a set with Property 1, consider the "final" elements, i.e. the elements where you can't follow any more arrows (forwards) without leaving the set.)

Question 7. Here are the pictures again.







Choose your favourite cube in the cubical complex. (It doesn't have to be a 3D cube!) Notice that it has a "biggest" vertex and a "smallest" vertex, in terms of the labels from the previous page.

(a) How many elements differ between the biggest and smallest vertices? How does that relate to the dimension of the cube?

(b) Consider the biggest and smallest vertex labels as subsets of the hyperplane relationship diagram. Where do the "differing" elements fit in to the "biggest" set?

Question 8. Let's finish this off! Here's what we observed:

- In the formula $(s+1)^n$, the coefficient of s^i is the number of subsets of size i in a set of size n.
- Vertices in the cubical complex $\longleftrightarrow$ sets of hyperplanes with Property 1
- Vertex sets of simplices in the crossing complex $\longleftrightarrow$ sets of hyperplanes with Property 2
- Sets with Property 1 $\longleftrightarrow$ sets with Property 2 (for a set with Property 1, take its "final" elements)
- To specify an i-dimensional cube in the cubical complex, we can pick a "biggest" and "smallest" vertex for that cube, and the "differing" elements have to be i "final" elements of the hyperplane set for the bigger vertex.

Put these pieces together:

(a) Explain why the number of i-dimensional faces of the cubical complex equals the number of ways to pick a set with Property 2 and a subset of this set of size i.

(b) Explain why this number equals the coefficient of s^i in

$$1 + (\#\text{vertices}) \cdot (s+1) + (\#\text{edges}) \cdot (s+1)^2 + (\#\text{triangles}) \cdot (s+1)^3 + (\#\text{tetrahedrons}) \cdot (s+1)^4$$

Question 9. *(Challenge.)* Draw a relationship diagram with only dotted lines, no arrows. Then, try to build a cubical complex that has this hyperplane relationship diagram.

Can you figure out a way to do this for *any* arrow-less relationship diagram?

What can you say about the position of the "home" vertex?