Week 3

Question 1. You're the coach of a football team, with four players: Anna, Bruce, Conrad and Delores. You need to pick two of the players to form a special sub-team.

Write down all possible choices of two players.

How many possibilities are there?

Question 2. What if you had to pick a different number of players for the sub-team? Fill in this table:

Size of sub-team	1	2	3	4
Number of possibilities				

If you were at Math Circle last week: do these numbers look familiar. . . ?

Question 3. Your team just recruited a new member, Egbert, so now there are five players to choose from. Now, how many ways are there to choose subteams?

Size of sub-team	1	2	3	4	5
Number of possibilities					

Question 4. Last week, we studied Pascal's Triangle, which looks like this:

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&&\\
&&&&&&&1&&1&&&&&&&\\
&&&&&&1&&2&&1&&&&&&\\
&&&&&1&&3&&3&&1&&&&&\\
&&&&1&&4&&6&&4&&1&&&&\\
&&&1&&5&&10&&10&&5&&1&&&\\
&&1&&6&&15&&20&&15&&6&&1&&\\
&1&&7&&21&&35&&35&&21&&7&&1&\\
1&&8&&28&&56&&70&&56&&28&&8&&1\\
&\vdots&&\vdots&&\vdots&&\vdots&&\vdots&&\vdots&&\vdots&&\vdots&
\end{array}
$$

Based on this triangle and your observations on the previous page, guess how many ways there are to choose:

(a) a sub-team of size 2 from a football team of 7 players?

(b) a sub-team of size 4 from a team of 8 players?

These numbers of ways to choose sub-teams come up a lot in math, and they have a special name: they're called "binomial coefficients". There's also a special notation:

$$\binom{n}{k} \qquad \text{means} \qquad \text{"the number of ways to choose } k \text{ things out of a set of } n \text{ things".}$$

Question 5. To make sure you understand the notation, fill in the blanks:

(a) $\binom{4}{2}$ = number of ways to choose ______ things out of a set of ______ = ______

(b) $\binom{5}{3}$ = number of ways to choose ______ things out of a set of ______ = ______

(c) $\binom{6}{1}$ = number of ways to choose ______ things out of a set of ______ = ______

Question 6. What number is $\binom{n}{n}$, and why? What about $\binom{n}{1}$ and $\binom{n}{n-1}$?

Question 7. What number do you think $\binom{n}{0}$ should be? What about $\binom{n}{-1}$ and $\binom{n}{n+1}$? Does this make any sense?

Question 8. Last time we noticed some patterns in Pascal's Triangle. For example, the diagonal that's two levels in from the edge contains this sequence of numbers:

$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ \ldots$$

The nth number in this sequence is always n more than the previous one. Translated into binomial coefficients, this statement becomes:

$$\binom{n+1}{2} = \binom{n}{2} + n.$$

Why does this work?

(Hint: If we add one more player to our team, how many ***new*** *sub-teams of size* 2 *can we make?)*

Question 9. The rule for building Pascal's Triangle is "each number is the sum of the two numbers above it". In the language of binomial coefficients, this rule is:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Why is this true?

(Hint: How many teams include Anna?)

Question 10. Another observation about Pascal's Triangle is that it's symmetric. If we were to state this fact in terms of binomial coefficients, we could write it like this:

$$\binom{n}{k} = \binom{n}{n-k}$$

(i.e. "the kth number from the left in the nth row is the same as the kth number from the right in the nth row").

Explain why this fact is true, in terms of choosing subsets.

(Hint: Try thinking about the $k = 1$ case first.)

Question 11. One other pattern we found in Pascal's Triangle was that if you add up all the numbers in the nth row, you get 2^n. In terms of binomial coefficients:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

Why should this be true?

Question 12. Here's one other fact about Pascal's Triangle: in each row, if you add up every second entry, you get the same answer as if you add up the other entries. In binomial coefficients:

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

Explain why this fact is true.