UW Math Circle
April 2, 2020

Number theory problems

1. Consider the Fibonacci numbers mod 6: $1, 1, 2, 3, 5, 2, 1, 3, \ldots$ Does this sequence ever repeat? What about for mods other than 6?

2. Compute the powers of 3 mod 16: $3, 3^2 \equiv 9, 9 \cdot 3 \equiv 11, \ldots$. Do the same with powers of 5 mod 17, or other values. Does the sequence always eventually repeat? If it does, how long does it take to repeat?

3. Find all the numbers mod 8 that have an inverse mod 8. (For example, 2 has no inverse mod 8, because there is no number x such that $2x \equiv 1 \mod 8$.) How many are there? Do the same for mod 6 and mod 7. Try to prove any patterns you notice.

4. Without using a calculator, find the last two digits (in base 10) of:

$$3^{45} \qquad 7^{7^{7^{7^{7^{7^{7}}}}}} \qquad 1 \cdot 6 \cdot 11 \cdots 91 \cdot 96 \cdot 101$$

5. Each of these equations has exactly four solutions. Find them and prove that there are no other solutions.

$$x^2 \equiv 1 \mod 15 \qquad x^2 \equiv 1 \mod 35 \qquad x^2 \equiv 25 \mod 1739$$

6. How many trailing zeros does 100! have? Try to find a general formula for the largest power of p dividing n for a prime p.

7. Seven competitive math circle students try to share a huge hoard of stolen math books equally between themselves. Unfortunately, six books are left over, and in the fight over them, one math student is expelled. The remaining six students, still unable to share the math books equally – this time two are left over – again fight, and another is expelled. When the remaining five share the books, one book is left over, and it is only after yet another math student is expelled that an equal sharing is possible. What is the minimum number of books they could have started with?

8. Let n be a positive integer such that $n \equiv -1 \mod 24$, and let $S =$ sum of divisors of n. Show that $S \equiv 0 \mod 24$.

9. (From Putnam 1998) Define a sequence a_n as follows: $a_1 = 0, a_2 = 1$, and a_{n+2} is obtained from a_n and a_{n+1} by concatenating the strings of digits of a_{n+1} and a_n. For example, $a_3 = 10, a_4 = 101, a_5 = 10110$. Show that $11 | a_n$ if and only if $n \equiv 1 \mod 6$.

10. Suppose p is a prime number. Show that $p^2 + 2$ is not prime unless $p = 3$.

11. Show that if p is prime, then $(a + b)^p \equiv a^p + b^p \mod p$.

12. Prove that there are infinitely many primes p such that $p \equiv 3 \mod 4$.