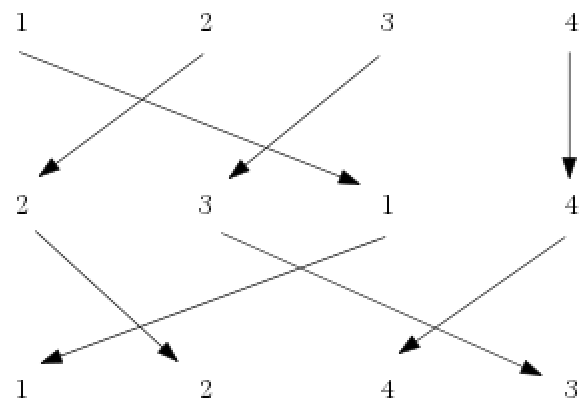
UW Math Circle, Fall 2018

A tune, import

Given two permutations A and B of the same length, the permutation $A \times B$ is defined by first applying permutation A to the 'identity' permutation $1\ 2\ 3 \cdots n$, then applying B to the result. For example, if and $A = 3124$ and $B = 2413$, then $A \times B$ can be visualized by placing A 'on top' of B and following the arrows:



So $A \times B = 1243$.

Alternatively, we can use cycle notation: $A = (132)(4), B = (1243)$. The cycles are read left to right, with 'wrap around': A sends 1 to 3, 3 to 2, 2 to 1, and 4 to itself. Multiplying just means putting the cycles next to eachother and following the numbers around:

$$A \times B = (132)(4)(1243) = (1)(2)(34). \tag{1}$$

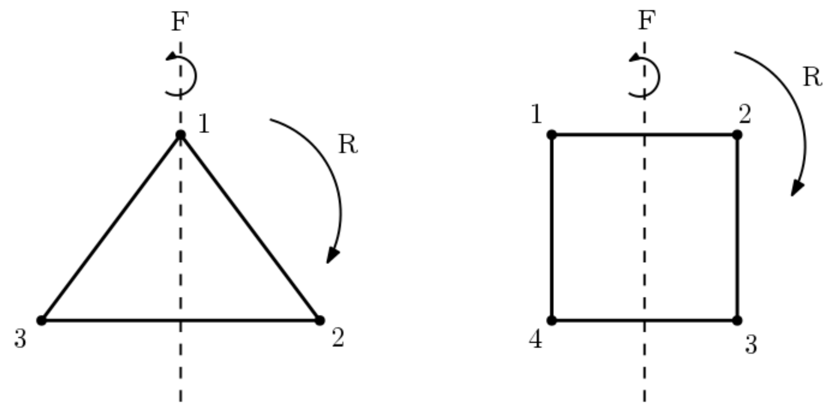
1. Why does cycle notation always work, so that you end up with a permutation?
2. Why do these two ways of multiplying always give the same answer?

Give the cycle notation for the following:

- $(1234)(4321) =$ ________________
- $(12)(13)(14) =$ ________________
- $(12)(23)(34) =$ ________________
- $(123)(234)(341)(412) =$ ________________

3. (Inverses) Given a permutation A, is there a permutation A^{-1} with $A \times A^{-1} = I$ (where I is the identity permutation $I = (1)(2)(3)\cdots)$? How can you construct such an A^{-1} from A? Is it possible to have multiple permutations for A^{-1}?
4. (Commutativity and associativity) Is it true that $A \times B = B \times A$ for permutations A and B? What about $A \times (B \times C) = (A \times B) \times C$ for permutations A, B, C?
5. (Powers) What happens to the sequence $A, A \times A, A \times A \times A, \ldots$ for a permutation A? Does it ever repeat?
6. (Square roots) Does the permutation $A = (123)$ have a square root, i.e. a permutation $\sqrt{A}$ with $\sqrt{A} \times \sqrt{A} = A$? How many square roots does it have? What about $B = (1234)$ and $C = (12)(34)$?
7. (Generation) How many permutations do you need to get all the permutations of length 4 by multiplying? Find such a set of permutations that is as small as possible.

Last class, we talked about symmetries of the triangle and the square, using flips and rotations.



We can think of the numbers in our permutations as the vertices of the triangle or the square, which are being permuted by flipping or rotating.

8. What permutation represents F, the flip operation for the triangle? For the square?

9. What permutation represents R, the rotation operation for the triangle? For the square?

10. We noticed last class that for the triangle, flipping and then rotating twice is the same as rotating and then flipping. Prove this using permutations.

11. Are there are any similar identities for the square? Try to find one, and prove it using permutations.