Divisibility tricks

Last week, we introduced *modular arithmetic*: we said that "$a \equiv b \pmod{q}$" (or "a is congruent to b modulo q") if a and b have the same remainder when you divide them by q. Remember:

$$\text{if} \quad a \equiv a' \pmod{q} \quad \text{and} \quad b \equiv b' \pmod{q}, \qquad \text{then} \quad a + b \equiv a' + b' \pmod{q}, \quad \text{and}$$
$$\text{if} \quad a \equiv a' \pmod{q} \quad \text{and} \quad b \equiv b' \pmod{q}, \qquad \text{then} \quad a \times b \equiv a' \times b' \pmod{q}.$$

Today we're going to use these facts to explain some divisibility tricks!

Divisible by 3?

Imagine you want to know if 62 947 065 is divisible by 3. How can you check this? Many of you probably know an easy way — add the digits together, then divide *that* number by 3 — but let's see why this works.

Well,

$$\begin{aligned} 62\,947\,065 &= 60\,000\,000 + 2\,000\,000 + 900\,000 + 40\,000 + 7\,000 + 60 + 5 \\ &= 6 \times 10\,000\,000 + 2 \times 1\,000\,000 + 9 \times 100\,000 + 4 \times 10\,000 + 7 \times 1\,000 + 6 \times 10 + 5 \times 1. \end{aligned}$$

What is this congruent to modulo 3? What are $10\,000\,000, 1\,000\,000, 100\,000$ etc. congruent to modulo 3, and why? (*Hint:* $100\,000 = 10 \times 10 \times 10 \times 10 \times 10$.)

Divisible by 9?

Can you use the same ideas to check if a number is divisible by 9?

Divisible by 11?

If you want to check if a number's divisible by 11, you can alternate adding and subtracting its digits (starting by adding the rightmost one). For example, 2 876 412 is divisible by 11 since $2 - 1 + 4 - 6 + 7 - 8 + 2 = 0$. Why does this work? (*Hint:* $10 \equiv -1 \pmod{11}$.)

Other numbers?

Can you come up with any more divisibility tricks? Try 4, or 6, or 99, or 37, or 13.

What about other bases? For example, how can you check if a number in hexadecimal is divisible by 5? Which binary numbers are divisible by 3?