Euler's formula: Polytopes

Here are some polytopes:

A *polytope* is a 3D shape whose sides (or "faces") are polygons (squares, triangles, hexagons etc. — they don't have to be regular). Can you think of any more polytopes?

Problem 1. How many polytopes can you find where:

- . . . all faces are triangles? Quadrilaterals? Pentagons? Hexagons?

- . . . every corner of the shape is in exactly 3 faces? Or 4? Or 5, or 6. . . ?

Problem 2. For each polytope, count the numbers of corners, edges and faces. Do you notice any patterns? *(Hint: Try adding the numbers of corners and faces together.)*

Euler's formula: Planar graphs

Remember that a graph is *planar* if you can draw it (in the 2D plane) without any of the edges crossing over each other. Here are some planar graphs:

Can you draw some more?

Problem 3. Try to find some planar graphs where:

- ...all vertices are contained in exactly 3 edges? Or 4, or 5, or 6,...
- ...every region between the edges has 3 sides? Or 4, 5, etc....

Problem 4. For each planar graph, count the numbers of vertices, edges and regions[1]. Do you notice any patterns?

Problem 5. What's the connection with polytopes?

[1]Question: Should you count the "outside" of the graph as a region? What do you think?

Euler's formula: Why is it true?!

On the last worksheet, hopefully you discovered this fact: if you add the numbers of vertices and regions in a (connected) planar graph, then subtract the number of regions, you always get 1 (or 2, depending on whether the "outside" counts as a region). In other words, $V - E + F = 1$, where V, E and F stand for the numbers of vertices, edges and regions respectively. In this worksheet, we're going to figure out why this works!

To begin with, let's think about trees. A *tree* is a graph that doesn't have any cycles — that is, it's impossible to follow the edges and get back to where you started without retracing your steps.

Problem 6. Explain why the following fact is true: In every tree that has at least 2 vertices, there is some vertex which only touches one edge. (Such a vertex is called a *leaf*.)

(Hint: Try walking along edges without retracing your steps.)

Problem 7. Now, explain why Euler's formula works for trees.[2]

(Hint: First, explain why it works for trees with 2 vertices or fewer. Next, if you have a tree with at least 2 vertices, is there something you can do to turn it into a smaller tree?)

Problem 8. Finally, explain Euler's formula for all planar graphs.

(Hint: If a graph is not a tree, it must contain a cycle. Now, pick an edge in that cycle, and try something similar to what you did in problem 7.)

[2] Question: Is it true that every tree is a *planar* graph?

Euler's formula: What is it good for?

Now let's try to use Euler's formula to prove some stuff!

Problem 9. In problem 1, you probably found it difficult to make any polytopes using only hexagons. Let's figure out why: we'll imagine that we've found a polytope using only hexagons, and we'll try to write down Euler's formula for this polytope.

(a) Use the facts that each face has 6 edges and each edge is contained in 2 faces to write an equation relating E and F.

(b) Now, each face has 6 corners, and each corner is contained in at least 3 faces. (If you like, you can assume that all corners are in *exactly* 3 faces — it'll make the calculations simpler.) Write down an equation relating V and F.

(c) Finally, substitute all these equations into Euler's formula. What happens?

Problem 10. Making a polytope using just hexagons is rather hard, so let's throw in some pentagons as well! Let P stand for the number of pentagons, and H the number of hexagons. Try to do some similar calculations to problem 9. What can you say about the number of pentagons you need?