Graph colourings

Remember from last week:

- A *graph* is a picture with some dots (called "vertices") and some lines (called "edges") connecting some of the dots.
- A *colouring* of a graph is a choice of a colour for each of the vertices (e.g. red, purple, grey,...), so that every edge connects two vertices with *different* colours. The smallest number of colours you need to colour a graph is called the *chromatic number* of the graph.

Problem 1. How many colours do you need to colour these graphs:

(a) a triangle, a square, a pentagon, a hexagon,...

(b) a path graph (i.e. a graph where all the edges line up in a straight line) with 1 edge, or 2 edges, or 3, or 4,...

(c) a complete graph (i.e. a graph where *every* pair of vertices is connected by an edge) with 3 vertices, or 4 vertices, or 5, or 6,...

Problem 2. Try to find as many graphs as you can whose chromatic number is 1, or 2, or 3, or 4.

Problem 3. A *clique* in a graph is a set of some of the vertices of the graph where every pair of vertices in the set is connected by an edge. For example, in a complete graph, the set of all vertices is a clique, and in a path graph, any two adjacent vertices are a clique. What is the relationship between the chromatic number of a graph and the size of the largest clique?

Problem 4. How many ways are there of colouring the following graphs with: 1 colour, 2 colours, 3 colours, 4 colours,...? Can you come up with a general formula for the number of ways of colouring them with n colours?

What numbers do you get if you set $n = (-1)$ in this formula?

Problem 5. A *directed graph* is like an ordinary graph except that the edges are arrows, not just lines. A directed graph is *acyclic* if you can't follow the arrows in a loop to get back to where you started. For example, these graphs are directed and acyclic:

How many ways can you turn the graphs above into acyclic directed graphs, by choosing directions for the arrows?