$\sqrt{2}$ is irrational!

We want to explain why $\sqrt{2}$ is an irrational number (that is, it isn't a fraction). Let's do this by imagining that $\sqrt{2}$ is rational, and seeing that this leads us to problems.

If $\sqrt{2}$ is rational, we can write it as a fraction:

$$\sqrt{2} = \frac{p}{q}.$$

If p and q have a common factor, we can cancel it — for example, $\frac{5}{10}$ is the same as $\frac{1}{2}$ after we divide on top and bottom by 5 — so let's also assume that p and q have no common factors. Now, we'll get rid of the $\sqrt{\ }$ by squaring everything:

$$\left(\sqrt{2}\right)^2 = \left(\frac{p}{q}\right)^2$$

so

$$2 = \frac{p^2}{q^2},$$

and then let's multiply both sides by q^2:

$$2q^2 = p^2.$$

Problem 1. I claim that this means p must be an even number. Why?

Since p is even, it must be 2 times some other number: that is, $p = 2r$ for some r. Plugging this into our equation, we get:

$$\begin{aligned} 2q^2 &= (2r)^2 \\ &= 2^2 r^2 \\ &= 4r^2. \end{aligned}$$

And now let's divide both sides of the equation by 2:

$$q^2 = 2r^2.$$

Problem 2. What does this tell us about q?

Problem 3. Finish this explanation off by explaining why this is impossible.

Problem 4. Can you explain why $\sqrt{3}$ is irrational? What about $\sqrt{4}$?