Collapsing compass

Last week we had a lively discussion on whether you should be allowed to pick up your compass and put it down somewhere else while keeping the same radius. A compass that can't do that is called a "collapsing compass".

I claim that this rule doesn't matter — even if you only have a collapsing compass, you can still construct all the same pictures as with a regular compass! Let's see why.

Problem 1. We'll start with something we saw last week. Can you find the midpoint of a line segment using only a ruler and *collapsing* compass?

Problem 2. Let's start getting serious. Suppose you're given a line, with two marked points on the line that are 1 inch apart. If I pick any other point on that line, can you mark another point that's 1 inch further away along the line, using just a ruler and collapsing compass?

Problem 3. Now, suppose I give you a line segment that's 1 inch long, and some other point. Can you draw another point that's 1 inch away from the first one?

(Hint: First, try to turn this picture into the picture from problem 2.)

Explain why this means that the collapsing compass is just as powerful as the regular one.

Short ruler

So far, we've been assuming that our ruler is infinitely long, and you can connect points that are as far apart as you like. Now, I don't know about you, but all of my rulers are only 12 inches long, not ∞ inches. Does this matter?

Problem 4. If you only have a 12-inch ruler and a compass, can you still draw a straight line between two points that are 13 inches apart? (For now, you can assume that your compass is infinite-sized.)

(Hint: You'll need to draw the line in two roughly 7-inch pieces, or something like that, but how do you make sure the two pieces line up?)

Problem 5. Now, can you draw a straight line between points that are 15 inches apart? Or 25 inches? 50 inches? A mile?

Problem 6. What if your compass can only reach up to 12 inches too?

Ruler or compass?

Problem 7. Imagine that you disproved another mathematician's conjecture, and they stole your ruler in revenge. Can you still construct anything with just your compass?

(a) Given a circle and its centre, can you mark three points on the circle that divide the circumference into three pieces of equal length? Can you do the same but with six pieces instead of three? What about four pieces?

(b) What if you don't know where the centre of the circle is? Can you find it?

Problem 8. Your mathematical enemy now has a ruler, but no compass. Can you help them construct some things using just a ruler?

Suppose your enemy is given two parallel lines and a point. Can they construct a third parallel line passing through the point?

Caliper

An ordinary compass has one pencil end and one spiky end. If you replace the pencil with another spike, you get a thing called a "caliper". This tool can't be used to draw circles, but if you have a point on a line, you can use a caliper to find another point on the line that's a certain distance from the first one.

(You can pick up a caliper without losing the radius — it's not like the collapsing compass.)

Problem 9. If you're given a line segment of length 1 inch, can you use a ruler and caliper to make this line segment twice as long? Or half as long (i.e. find its midpoint)?

Note: You can't use the usual trick of drawing two circles with the same radius and then seeing where they intersect, since you can't draw circles any more!

Problem 10. If you're given a line segment of length 1 inch, can you turn it into a 1-inch by 1-inch square?

Goniometer

A *goniometer* is a tool for drawing angles. It's basically just two rulers attached with a pivot — if you line up the two rulers with the two lines of an angle, you can then copy that angle somewhere else. Some fancy goniometers let you set the two arms to specific angles, but you just have a cheap one, so you can only set it to angles you've already constructed.

Problem 11. If I give you two lines meeting at a 60° angle, can you construct a 90° angle, using just a ruler and goniometer? Can you do this in reverse — can you make a 60° angle out of a 90° one?

Problem 12. If I give you a 45° angle, can you construct a square? A regular octagon?

Problem 13. Can you construct a 90° angle starting from a blank piece of paper?

Matchsticks

In this question, you don't have a ruler or compass, but you do have a box full of matchsticks. Each matchstick is exactly 1 inch long, and extremely thin, so you can think of them as a bunch of 1-inch line segments. You can't draw circles any more, and you can only draw lines between points that are 1 inch apart or closer. However, there's one special thing you can do with matchsticks: if you have a line segment that's less than 2 inches long, you can balance the ends of two matchsticks on the ends of the line segment, and bring the two free ends of the matchsticks together to make an isosceles triangle with two 1-inch sides.

Problem 14. Can you make an equilateral triangle out of matchsticks? What about a regular hexagon?

Problem 15. If you have a line segment that's less than 1 inch long, can you find its midpoint?

Problem 16. Can you make a square?