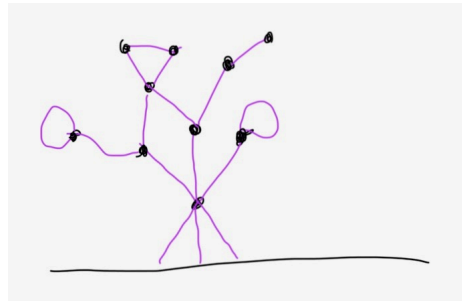
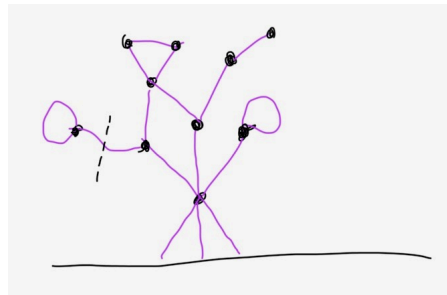


## Hackenbush

Violet and Dash are playing a game called “Hackenbush”. Their garden has this small bush in it:



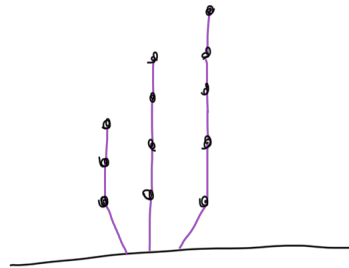
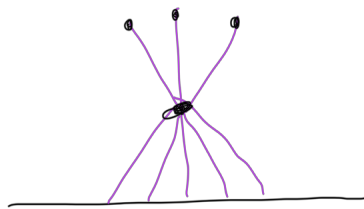
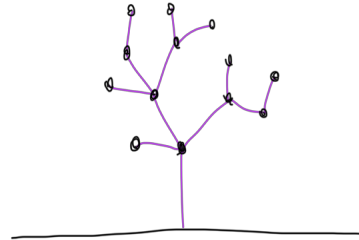
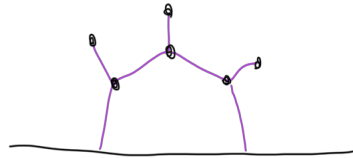
And the siblings are going to hack it apart. They each take it in turns to cut one branch of the bush, by deleting an edge in the graph as well as anything that isn't attached to the bush any more. For example, Violet could start by cutting like this:



Then the cut edge and the loop at the end of it would disappear. If someone can't cut anything (because the whole bush is gone), they lose the game.

Find a partner, and try playing Hackenbush yourselves!

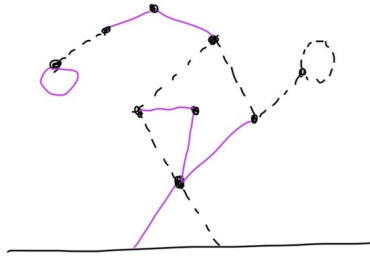
Here are some more bushes:



Does either player have a winning strategy for any of these bushes?  
Try making your own bushes to play Hackenbush on!

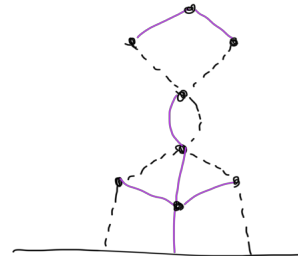
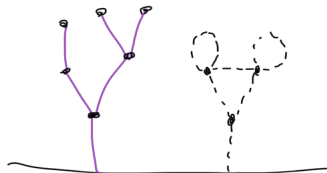
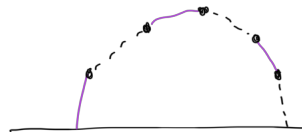
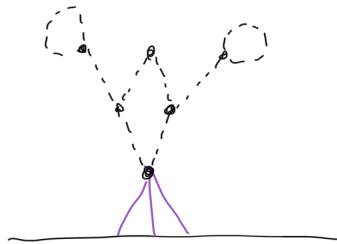
## Multicoloured Hackenbush

Dash and Violet ran out of bushes in their own garden, so they snuck into their neighbour's garden instead. Their neighbour is a keen gardener, so she grows more exotic bushes, like this one:



To make the game more interesting, Dash and Violet introduce a new rule: Dash is only allowed to cut the dashed edges, and Violet is only allowed to cut violet edges! (Well, grey edges. Colour printing is expensive.) Again, if either player can't make a move, they lose.

Here are some more bushes:



Does anyone have a winning strategy? Does it depend on who goes first?



## Hackenbush numbers

Dash is fed up with losing, so he decides to do some math instead. He decides to assign a number (positive or negative) to each bush, to indicate how favourable it is for him.

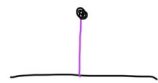
Let's start with something simple:



Who wins if Dash goes first? If Violet goes first?

Dash likes this bush, so he decides to assign it a value of 1.

What number should this game get?



Who wins this game? Does it matter who goes first?



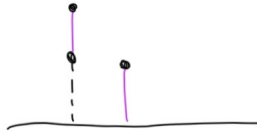
Dash decides to give this game the value 0. How does this compare with the previous two bushes?

It looks like the operation “put two bushes next to each other” matches up with the operation “adding” for their associated numbers. We’ll use that to figure out the value of this mystery bush:



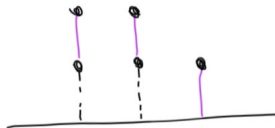
Is this a positive or negative bush, from Dash’s perspective?

What if we “add”  $-1$  to it:



Is this bush positive or negative?

It looks like the value of the mystery bush is positive, but less than 1. Now consider this bush:



What can you conclude about the mystery bush’s value?

Try growing some other simple bushes and figuring out their numbers!