Problem 1. The Fibonacci numbers are the sequence of numbers starting with $1, 1$ where each number is the sum of the two previous ones:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Try dividing each number in the sequence by the one before it. Do you notice any patterns?

The Lucas numbers are another Fibonacci-type sequence. They're defined the same way, except that they start with $2, 1$ instead of $1, 1$:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$$

Find ratios of pairs of Lucas numbers, like you did for the Fibonacci numbers. Any patterns here?

Now make your own Fibonacci-type sequence, by picking two starting numbers and filling in the rest of the sequence by adding the two previous numbers together at each step. Are there patterns in the ratios? Does it depend on what starting numbers you choose? Don't just use whole numbers!

Problem 2. A *geometric sequence* is one where the ratios between consecutive numbers are always exactly the same. For example, in this sequence, the ratio is always 2:

$$1, 2, 4, 8, 16, 32, 64, \ldots$$

Is there a sequence of numbers that is both a geometric sequence and a Fibonacci-type sequence? If so, what can you say about the ratio, or the starting numbers?

Problem 3. There's a special number that kept coming up in the previous worksheet. It's $\phi = 1.618033989\ldots = \frac{1+\sqrt{5}}{2}$, called the "Golden Ratio" — you might have heard of it. One interesting feature of this number is that it's close to a lot of fractions. To see what I mean, let's talk about *continued fractions.*

We'll start with an easier example — let's take the number $\frac{86}{27}$. This number is a little bit more than 3, so let's write it as

$$\frac{86}{27} = 3 + \frac{5}{27} = 3 + \frac{1}{\frac{27}{5}}.$$

Now, $\frac{27}{5}$ is a little bit more than 5, so we can write

$$= 3 + \frac{1}{5 + \frac{2}{5}} = 3 + \frac{1}{5 + \frac{1}{\frac{5}{2}}}.$$

And $\frac{5}{2}$ is a little bit more than 2, so let's finish this off:

$$= 3 + \frac{1}{5 + \frac{1}{2+\frac{1}{2}}}.$$

It turns out that any fraction can be written like $a + \cfrac{1}{b + \frac{1}{c+\frac{1}{d+\cdots}}}$, in essentially a unique way. This kind of expression is called a "continued fraction".

Try it yourself for these numbers:

$$\frac{43}{30}, \qquad \frac{65}{29}, \qquad \frac{559}{504}, \qquad \frac{413}{134}, \qquad \frac{134}{413}, \qquad \frac{1807}{350}$$

Continued fractions take up a lot of space, so instead of writing $a + \cfrac{1}{b + \frac{1}{c + \frac{1}{d + \cdots}}}$, let's just write $[a, b, c, d, \ldots]$. For example, from the calculation I showed you on the previous page, $\frac{86}{27} = [3, 5, 2, 2]$.

Problem 4. So far, we've only found continuous fraction versions of numbers that were already fractions to start with. But we can do this for other numbers too! For example, π is a little bit more than 3:

$$\pi = 3 + 0.14159\ldots = 3 + \frac{1}{7.0625\ldots}$$

and $7.0625\ldots$ is a bit more than 7:

$$= 3 + \cfrac{1}{7 + \frac{1}{15.997\ldots}}$$

and so on:

$$= 3 + \cfrac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \cdots}}}} = [3, 7, 15, 1, 292, \ldots].$$

Now find the continued fraction version of the golden ratio.

Problem 5. What happens if you only take the first few numbers in the continued fraction? For π, for example, what happens if you just take $[3, 7, 15, 1]$, or $[3, 7, 15]$, or $[3, 7]$? It might help to see the pattern if you write these numbers as decimals. Try this for some of the other continued fractions you've calculated too.

Problem 6. Now, put problems 4 and 5 together. What can you tell me about fraction approximations of the golden ratio?