UW Math Circle (Ramsey thoery)

May 26th, 2016

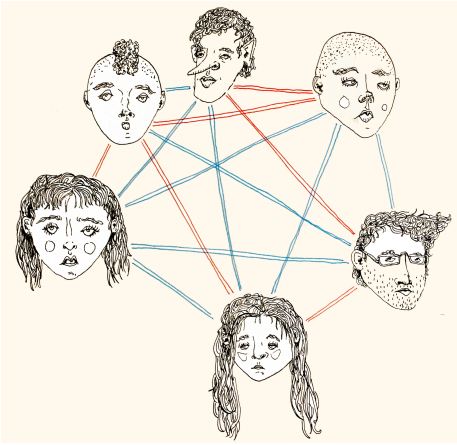
1 Ramsey numbers

"Suppose an evil alien would tell mankind 'Either you tell me the value of $R(5,5)$ or I will exterminate the human race.' ... It would be best in this case to try to compute it, both by mathematics and with a computer. If it would ask for the value of $R(6,6)$, the best thing would be to destroy him before he destroys us, because we couldn't.

— Paul Erdos

Define $R(m,n)$ to be the minimum integer N such that, if $N \geq R(m,n)$ and each edge of K_N is colored red or blue, then there must exist either a red K_m subgraph or a blue K_n subgraph. For example, $R(3,3) = 6$.

1) Given six people, show that either three are mutual friends, or three are complete strangers to one another. (Assume that "friendship" is mutual; i.e., if you are my friend then I must be your friend.)



2) Seventeen people are at a party. It turns out that for each pair of people present, exactly one of the following statements is always true: "They havent met," "They are good friends," or "They hate each other." Prove that there must be a trio (3) of people, all of whom are either mutual strangers, mutual good friends, or mutual enemies.

3) Ten people are in an elevator. Prove that either three know each other or four people are mutual strangers. Show that this is not necessarily true if we only have eight people.

4) Color the lattice points of the plane in two colors. Prove that there must be a rectangle (with sides parallel to the axes) each of whose vertices are the same color

Prove that (with appropriate conditions on m, n, r)

$$R(m,n) \leq R(m,n-1) + R(m-1,n).$$

Hint: induct on $m+n$.

Using the recurrence relations and constructions, show that

(a) $9 \leq R(3,4) \leq 10$.

(b) $14 \leq R(3,5) \leq 15$.

(c) $R(p,q) \leq \binom{p+q-2}{p-1}$