UW Math Circle

January 26th, 2017

1. Fill in the following blanks with the smallest non-negative integer satisfying the equation. (Remember, $x \equiv y(\mod n)$ if x and y have the same remainder when divided by n.)

   (a) $7 \equiv$ ___ mod 3

   (b) $-3 \equiv$ ___ mod 5

   (c) $2n+1 \equiv$ ___ mod 2

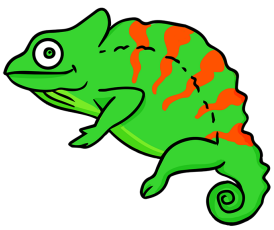
   (d) $2n+1 \equiv$ ___ mod n

2. What is the last digit of $1^2+2^2+3^2+\cdots+9^2$? What about $1^2+2^2+3^2+\cdots+99^2$?

3. Show that 855 cannot be written as a^2+b^2 where a and b are integers.

4. Let a, b, and c be the side lengths of a right triangle, and suppose they are all integers. Show that at least one of a, b, and c is divisible by 3, and at least one of a, b, and c is divisible by 5.

5. On a tropical island, far far away, there is a chameleon colony that lives by the following rules: the chameleons are three colors–either red, yellow, or blue–and whenever two chameleons of opposite color meet, they both change to the third color. So, if a red and yellow chameleon meet, they both become blue. If the island starts with 10 red chameleons, 11 yellow chameleons, and 12 blue chameleons, is it possible for them to eventually become all the same color?



1. Show that $n^5 - n$ is always divisible by 30.

2. Show that $n^9 + n^7 - n^3 - n$ is always divisible by 210.

3. Can 688 be written as $a^3 + b^3 + c^3$? What about 689?

4. Earlier, we had the problem : Show that 855 cannot be written as $a^2 + b^2$ where a and b are integers. Take a minute to remember your answer for that question!

   (a) Can 10015 be written as a sum of two squares?

   (b) Inspired by (a), develop a rule for deciding when an odd number cannot be written as a sum of two squares.

   (c) Similarly, develop a rule for deciding when an even number cannot be written as a sum of two squares.

   (d) Can every number that doesn't satisfy your rules for (b) and (c) be written as a sum of two squares? If not, try to find a way to describe all numbers that can be written as a sum of two squares.