# UW Math Circle

December 1, 2016

Homework

1. There are ten points marked on a plane so that no three of them are on the same straight line. How many triangles are there with vertices these 10 points?

2. Santa is wrapping 12 presents with either red, green, or silver wrapping paper. In how many ways can he do this?

3. In class, we found two different counting problems that had the same answer. Here, we will discover a nice formula to solve these counting problems.

   (a) The downtown of Cataland is a n by n grid of blocks (with streets in between). You are at the bottom leftmost intersection of downtown (so at the lower left hand corner of the n by n grid), and your hotel is at the top rightmost intersection of downtown (so at the upper right hand corner). You want to get to your hotel as fast as possible, so you will only travel up and to the right. However, there is a bad traffic jam in the upper left hand part of downtown Cataland, so you don't want to travel above the straight line connecting your present location with your hotel.

   How many paths can you take to your hotel when $n = 8$? Remember, you can only move up or to the right along the grid, and you can't go above the main diagonal.

   (b) How many paths are there for you to take to your hotel, if you don't care about the traffic jam? In other words, how many paths are there from the lower left hand corner of the grid to the upper right hand corner if you are only traveling up or two the left? Your answer should be in terms of n.

   (c) Now, given an arbitrary path as in part b, measure how far off it is from the type of path in part a by keeping track of how many vertical segments there are above the diagonal. Call this the "excess" of the path. Devise a reversible procedure for converting a path of excess x to excess $x - 1$. The picture below is a hint, where the excess changes from 3 to 2. The idea is that you can cut a path into two pieces, and rearrange them to lower the excess by exactly one.
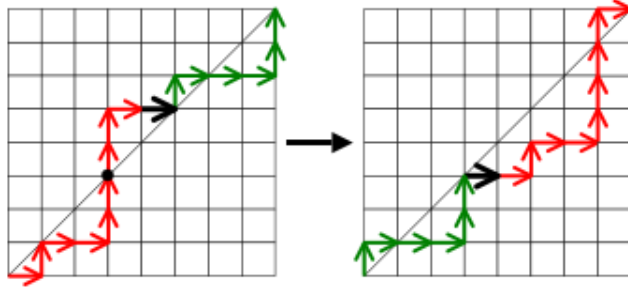
(d) Conclude that the number of paths of excess 0 is the same as the number of paths of excess 1, which is the same as the number of paths of excess 2, etc. Now using part b, figure out a formula for the number of paths to your hotel. Your formula should be in terms of n.

(e) Now, look at the worksheet from last week, and find the problems that this problem is the 'same as'. Your formula from part d can help you answer these questions much more easily now!